EECS 142

Two-Port Networks and Amplifiers

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1 Introduction to Two-Port Parameters

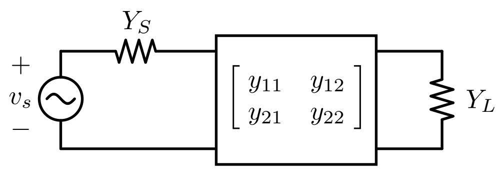


Figure 1: A generic amplifier represented as a two-port.

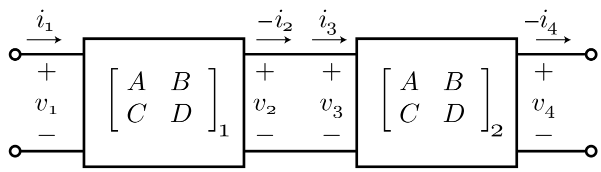


Figure 2: If we reverse the current direction on the second port, we can cascade two-ports using the $ABCD$ parameters.

Consider the generic two-port amplifier shown in Fig. 1. Note that any two-port linear and time-invariant circuit can be described in this way. We can use any two-port parameter set, including admittance parameters Y, impedance parameters Z, hybrid H or inverse-hybrid parameters G. These parameters represent a linear relation between the input/output voltages and currents. If we take linear combinations of current and voltage, we can derive other parameter sets, the most important of which is the scattering or S parameters. We may also choose to represent input versus output, which simplifies analysis of cascade of two-ports, such as the $ABCD$ parameter set

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_2 \\ -i_2 \end{pmatrix}$$

As shown in Fig. 2, the cascade of two blocks is obtained through simple matrix multiplication if we redefine the direction of i_2 so that it flows out of the first block and into the second block.

In this Chapter we review two-port parameters and derive equations for the gain, input/output impedance, and optimal source/load to realize the optimal gain. Next we introduce the important concept of scattering (S) parameters, which are used extensively in high frequency design of amplifiers, filters, and other building blocks. In the laboratory, we measure the properties of a circuit using a network analyzer, which measures the S parameters directly. While it is easy to convert from S parameters to other parameters, in many situations it will be convenient to "think" using s-parameters.

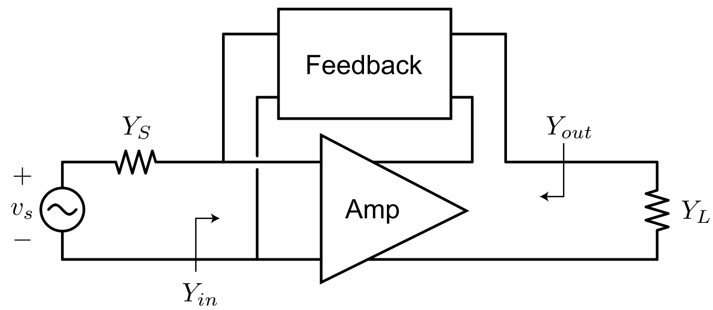


Figure 3: A generic feedback amplifier represented as an interconnection of two-ports. Note a series connection is made at the output (current sense) and shunted with the input (current feedback).

1.1 Choosing Two-Port Parameters

All two-port parameters are equivalent in their description of a linear system. The best choice of the parameter set is determined by finding the parameters that simplify calculations. For instance, if shunt feedback is applied, Y parameters are most convenient, whereas series feedback favors Z parameters. Other combinations of shunt/series can be easily described by H or G. In Fig. 3 the feedback is connected in series with the output and in shunt with the input so we see that we are sensing the output current and feeding back a current to the input. As such the most appropriate parameter set should involve currents/voltages which are the same for both blocks. In this case the input voltage and the output current are the same for each block whereas the total input current and output voltage are a summation of the amplifier and feedback blocks

$$\begin{pmatrix} i_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i_{a,1} \\ v_{a,2} \end{pmatrix} + \begin{pmatrix} i_{f,1} \\ v_{f,2} \end{pmatrix} = \begin{pmatrix} g_{11}^a & g_{12}^a \\ g_{21}^a & g_{22}^a \end{pmatrix} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} + \begin{pmatrix} g_{11}^f & g_{12}^f \\ g_{21}^f & g_{22}^f \end{pmatrix} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} g_{11}^a + g_{11}^f & g_{12}^a + g_{12}^f \\ g_{21}^a + g_{21}^f & g_{22}^a + g_{22}^f \end{pmatrix} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix}$$

As mentioned already, the $ABCD$ parameters are useful for cascading two-ports. Many of the results that we derive in terms of say Y-parameters can be applied to other two-port parameters (input impedance, output impedance, gain, etc) by simple substitution. In the laboratory we always use S parameters, since this is actually the way in which we measure two-port parameters at high frequencies.

1.2 Y Parameters

First let's use the Y or admittance parameters since they are familiar and easy to use

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Notice that y_{11} is the short circuit input admittance

$$y_{11} = \left. \frac{i_1}{v_1} \right|_{v_2=0}$$

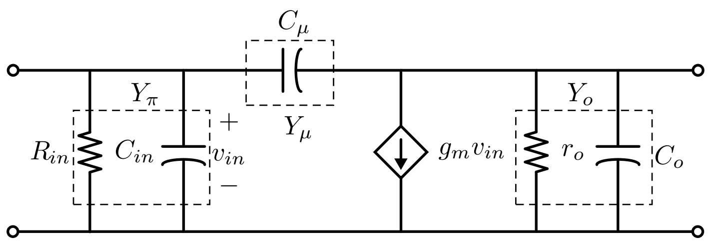


Figure 4: A hybrid-pi circuit as a two-port.

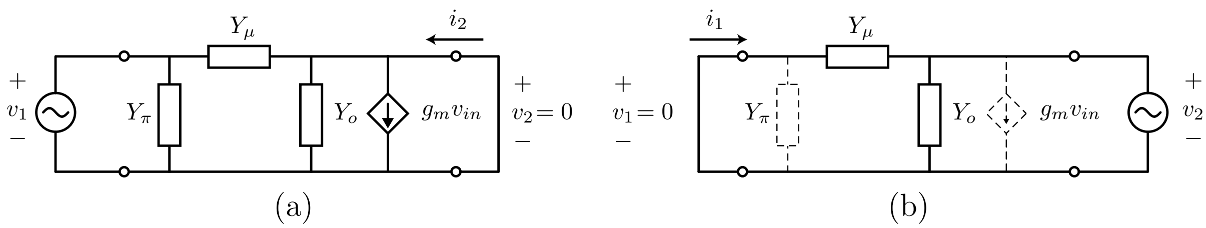


Figure 5: Setup to calculate (a) input admittance and (b) output admittance parameters.

The same can be said of y_{22}. The forward transconductance is described by y_{21}

$$y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0}$$

whereas the reverse transconductance is described by y_{12}. If a two-port amplifier is unilateral, then $y_{12} = 0$

1.3 Hybrid-Π Admittance Parameters

Let's compute the Y parameters for the common hybrid-Π model shown in Fig. 4. With the aid of Fig. 5a,

$$y_{11} = y_\pi + y_\mu$$

$$y_{21} = g_m - y_\mu$$

And with the aid of Fig. 5b

$$y_{22} = y_o + y_\mu$$

$$y_{12} = -y_\mu$$

Note that the hybrid-Π model is unilateral if $y_\mu = sC_\mu = 0$. Therefore it's unilateral at DC. A good amplifier has a high ratio y_{21}/y_{12} because we expect the forward transconductance to dominate the behavior of the device.

Why Use Two-Port Parameters?

Given that you can analyze amplifiers in detail using KVL/KCL, why use two-port parameters, which are more abstract than the equivalent circuit? The answer is that the parameters are generic and independent of the details of the amplifier. What resides inside the two-port can be a single transistor or a multi-stage amplifier. In addition, high frequency transistors are more easily described by two-port parameters (due to distributed input gate resistance and induced channel resistance). Also, feedback amplifiers can often be decomposed into an equivalent two-port unilateral amplifier and a two-port feedback section. Most importantly, two-port analysis will be used to make some very general conclusions about the stability and "optimal" power gain of a two-port. This in turn will allow us to define some useful metrics for transistors and amplifiers.

1.4 Voltage Gain and Input Admittance

Let's begin with some easy calculations for a loaded two-port shown in Fig. 1. Since $i_2 = -v_2 Y_L$, we can write

$$(y_{22} + Y_L)v_2 = -y_{21}v_1$$

Which leads to the "internal" two-port gain

$$A_v = \frac{v_2}{v_1} = \frac{-y_{21}}{y_{22} + Y_L}$$

The input admittance is easily calculated from the voltage gain

$$Y_{in} = \frac{i_1}{v_1} = y_{11} + y_{12}\frac{v_2}{v_1}$$

$$Y_{in} = y_{11} - \frac{y_{12}y_{21}}{y_{22} + Y_L}$$

By symmetry we can write down the output admittance by inspection

$$Y_{out} = y_{22} - \frac{y_{12}y_{21}}{y_{11} + Y_S}$$

For a unilateral amplifier $y_{12} = 0$ implies that

$$Y_{in} = y_{11}$$

$$Y_{out} = y_{22}$$

and so the input and output impedance are decoupled. This is a very important property of a unilateral amplifier which simplifies the analysis of optimal gain and stability considerably.

The external voltage gain, or the gain from the voltage source to the output can be derived by a simple voltage divider equation

$$A_v' = \frac{v_2}{v_s} = \frac{v_2}{v_1}\frac{v_1}{v_s} = A_v\frac{Y_S}{Y_{in} + Y_S} = \frac{-Y_S y_{21}}{(y_{22} + Y_L)(Y_S + Y_{in})}$$

If we substitute and simplify the above equation we have

$$A_v' = \frac{-Y_S y_{21}}{(Y_S + y_{11})(Y_L + y_{22}) - y_{12}y_{21}} \quad (1)$$

1.5 Feedback Amplifiers and Y-Params

Note that in an ideal feedback system, the amplifier is unilateral and the closed loop gain is given by

$$\frac{y}{x} = \frac{A}{1 + Af}$$

If we unilaterize the two-port by arbitrarily setting $y_{12} = 0$, from Eq. 1, we have an "open" loop forward gain of

$$A_{vu} = A'_v|_{y_{12}=0} = \frac{-Y_S y_{21}}{(Y_S + y_{11})(Y_L + y_{22})}$$

Rewriting the gain A'_v by dividing numerator and denominator by the factor $(Y_S + y_{11})(Y_L + y_{22})$ we have

$$A'_v = \frac{\frac{-Y_S y_{21}}{(Y_S+y_{11})(Y_L+y_{22})}}{1 - \frac{y_{12}y_{21}}{(Y_S+y_{11})(Y_L+y_{22})}}$$

We can now see that the "closed" loop gain with $y_{12} \neq 0$ is given by

$$A'_v = \frac{A_{vu}}{1 + T}$$

where T is identified as the loop gain

$$T = A_{vu} f = \frac{-y_{12}y_{21}}{(Y_S + y_{11})(Y_L + y_{22})}$$

Using the last equation also allows us to identify the feedback factor

$$f = \frac{y_{12}}{Y_S}$$

If we include the loading by the source Y_S, the input admittance of the amplifier is given by

$$Y_{in} = Y_S + y_{11} - \frac{y_{12}y_{21}}{Y_L + y_{22}}$$

Note that this can be re-written as

$$Y_{in} = (Y_S + y_{11}) \left(1 - \frac{y_{12}y_{21}}{(Y_S + y_{11})(Y_L + y_{22})} \right)$$

The last equation can be re-written as

$$Y_{in} = (Y_S + y_{11})(1 + T)$$

Since $Y_S + y_{11}$ is the input admittance of a unilateral amplifier, we can interpret the action of the feedback as raising the input admittance by a factor of $1 + T$. Likewise, the same analysis yields

$$Y_{out} = (Y_L + y_{22})(1 + T)$$

It's interesting to note that the same equations are valid for series feedback using Z parameters, in which case the action of the feedback is to boost the input and output impedance. For the hybrid H parameters, the action of the series feedback at the input also raises the input impedance but the action of the shunt output connection lowers the output impedance. The inverse applies for the inverse-hybrid G parameters.

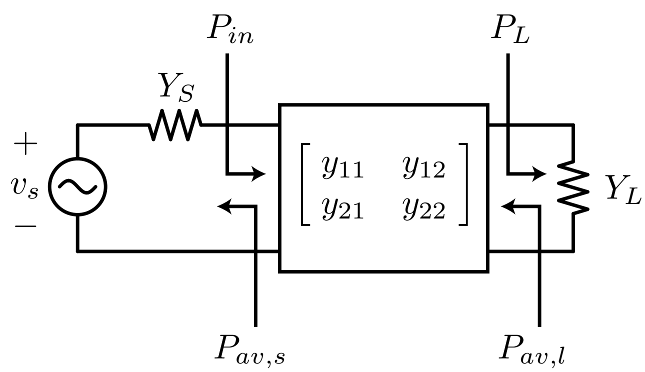


Figure 6: Various definitions of power in a two-port.

2 Power Gain

We can define power gain in many different ways. You may think that the *power gain* G_p is defined as follows

$$G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)$$

is the best way, but notice that this gain is a function of the load admittance Y_L and the two-port parameters Y_{ij}, but not the source admittance. In other words, G_p is the load power normalized by the input power. If the input power is very small, such as in a source mismatch condition, then the output power will also be small. This is hidden from G_p.

The *transducer gain* defined by

$$G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij})$$

measures the power deliverd to the load normalized by the available power from the source ($P_{av,S}$). This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match. As such it is a function of the source and the load.

The *available power gain* is defined as follows

$$G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L)$$

where the available power from the two-port is denoted $P_{av,L}$. This quantity is only a function of the load admittance and measures the efficiency of the output matching network.

The power gain is readily calculated from the input admittance and voltage gain

$$P_{in} = \frac{|V_1|^2}{2}\Re(Y_{in})$$

$$P_L = \frac{|V_2|^2}{2}\Re(Y_L)$$

$$G_p = \left|\frac{V_2}{V_1}\right|^2 \frac{\Re(Y_L)}{\Re(Y_{in})}$$

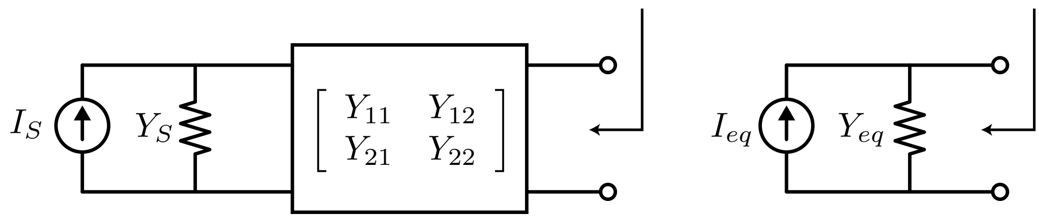


Figure 7: The Norton equivalent of a two-port from the output port.

$$G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})}$$

To derive the available power gain, consider a Norton equivalent for the two-port where (short port two) shown in Fig. 7

$$I_{eq} = I_2 = Y_{21} V_1 = \frac{Y_{21}}{Y_{11} + Y_S} I_S$$

The Norton equivalent admittance is simply the output admittance of the two-port

$$Y_{eq} = Y_{22} - \frac{Y_{21} Y_{12}}{Y_{11} + Y_S}$$

The available power at the source and load are given by

$$P_{av,S} = \frac{|I_S|^2}{8\Re(Y_S)}$$

$$P_{av,L} = \frac{|I_{eq}|^2}{8\Re(Y_{eq})}$$

$$G_a = \left|\frac{I_{eq}}{I_S}\right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

$$G_a = \left|\frac{Y_{21}}{Y_{11} + Y_S}\right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

The transducer gain is given by

$$G_T = \frac{P_L}{P_{av,S}} = \frac{\frac{1}{2}\Re(Y_L)|V_2|^2}{\frac{|I_S|^2}{8\Re(Y_S)}} = 4\Re(Y_L)\Re(Y_S)\left|\frac{V_2}{I_S}\right|^2$$

We need to find the output voltage in terms of the source current. Using the voltage gain we have and input admittance we have

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right|$$

$$I_S = V_1(Y_S + Y_{in})$$

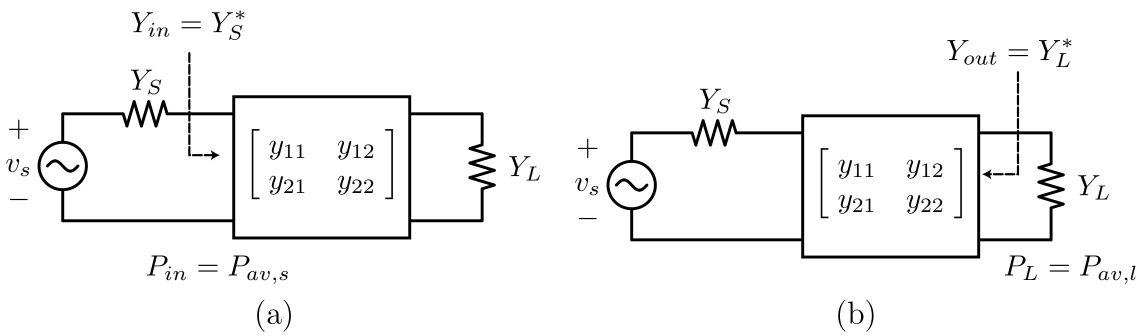


Figure 8: (a) A two-port matched at the input port. (b) A two-port matched at the output port.

$$\left|\frac{V_2}{I_S}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right| \frac{1}{|Y_S + Y_{in}|}$$

$$|Y_S + Y_{in}| = \left|Y_S + Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right|$$

We can now express the output voltage as a function of source current as

$$\left|\frac{V_2}{I_S}\right|^2 = \frac{|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

And thus the transducer gain

$$G_T = \frac{4\Re(Y_L)\Re(Y_S)|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

There is no need to redefine the power gains for the other parameter sets since *all* of the gain expression we have derived are in the exact same form for the impedance, hybrid, and inverse hybrid matrices. Simply change y to z, h or g.

2.1 Comparison of Power Gains

Since $P_{in} \leq P_{av,s}$, we see that $G_T \leq G_p$. Under what condition is $G_T = G_p$? Simply when the input impedance is conjugately matches to the source impedance (Fig. 8a). Since $P_L \leq P_{av,l}$, we see that $G_T \leq G_a$. Again, equality is obtained when the load is conjugately matched to the two-port output impedance (Fig. 8b). In summary

$$G_{T,max,L} = \frac{P_L(Y_L = Y_{out}^*)}{P_{av,S}} = G_a$$

$$G_{T,max,S} = G_T(Y_{in} = Y_S^*) = G_p$$

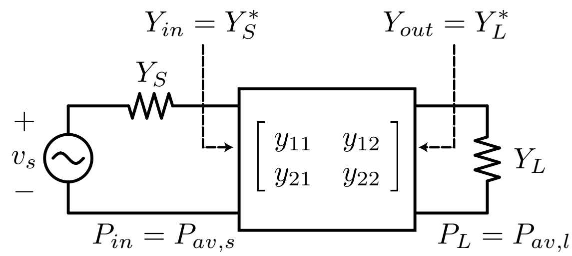


Figure 9: The bi-conjugate match, or simultaneous input and output match.

Input and Output Conjugate Match

It should be clear now that if we simultaneously conjugate match *both* the input and output of a two-port, we'll obtain the maximum possible power gain (Fig. 9). Under this condition all three gains are equal

$$G_{p,max} = G_{T,max} = G_{a,max}$$

This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} = Y_S^*$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}} = Y_L^*$$

Solution of the above four equations (real/imag) results in the optimal $Y_{S,opt}$ and $Y_{L,opt}$, or the solution to a pair of quadratic equations.

Calculation of Optimal Source/Load

Another approach to the problem of calculating the optimal source/load is to simply equate the partial derivatives of G_T with respect to the source/load admittance to zero to

$$\frac{\partial G_T}{\partial G_S} = \frac{\partial G_T}{\partial B_S} = \frac{\partial G_T}{\partial G_L} = \frac{\partial G_T}{\partial B_L} = 0$$

Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since G_a and G_p are only a function of the source or load, we can get away with only solving two equations. For instance

$$\frac{\partial G_a}{\partial G_S} = \frac{\partial G_a}{\partial B_S} = 0$$

This yields $Y_{S,opt}$ and by setting $Y_L = Y_{out}^*$ we can find the $Y_{L,opt}$. Likewise we can also solve

$$\frac{\partial G_p}{\partial G_L} = \frac{\partial G_p}{\partial B_L} = 0$$

And now use $Y_{S,opt} = Y_{in}^*$. Let's outline the procedure for the optimal power gain. We'll use the power gain G_p and take partials with respect to the load. Let

$$Y_{jk} = m_{jk} + jn_{jk}$$

$$Y_L = G_L + jX_L$$

$$Y_{12}Y_{21} = P + jQ = Le^{j\phi}$$

$$G_p = \frac{|Y_{21}|^2}{D} G_L$$

$$\Re\left(Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2}$$

$$D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22})$$

$$\frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2} \frac{\partial D}{\partial B_L}$$

Solving the above equation we arrive at the following solution

$$B_{L,opt} = \frac{Q}{2m_{11}} - n_{22}$$

In a similar fashion, solving for the optimal load conductance

$$G_{L,opt} = \frac{1}{2m_{11}} \sqrt{(2m_{11}m_{22} - P)^2 - L^2}$$

If we substitute these values into the equation for G_p (lot's of algebra ...), we arrive at

$$G_{p,max} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}}$$

Notice that for the solution to exists, G_L must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

The condition on the factor K is important as we will later show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of K

$$Y_{S,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L,opt} = \frac{Y_{12}Y_{21} + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p,max} = G_{T,max} = G_{a,max} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$

2.2 Maximum Gain

The maximum gain is usually written in the following insightful form

$$G_{max} = \frac{Y_{21}}{Y_{12}}(K - \sqrt{K^2 - 1})$$

For a reciprocal network, such as a passive element, $Y_{12} = Y_{21}$ and thus the maximum gain is given by the second factor

$$G_{r,max} = K - \sqrt{K^2 - 1}$$

Since $K > 1$, $|G_{r,max}| < 1$. The reciprocal gain factor is known as the efficiency of the reciprocal network. The first factor, on the other hand, is a measure of the non-reciprocity.

The case of a unilateral amplifier is of particular interest

$$G_{TU} = \frac{4|y_{21}|^2 \Re(Y_L)\Re(Y_S)}{|(y_{22} + Y_L)(Y_S + Y_{in})|^2}$$

The transducer gain is maximum under a conjugate input/output match

$$Y_S = Y_{in}^* = Y_{11}^*$$

$$Y_L = Y_{out}^* = Y_{22}^*$$

Resulting in a maximum unilateral gain

$$G_{TU,max} = \frac{|y_{21}|^2}{4\Re(Y_L)\Re(Y_S)}$$

Take for instance the hybrid-π model discussed earlier (Fig. 24). If we assume the model is unilateral, e.g. $C_\mu \approx 0$, then

$$y_{11} = Y_\pi + Y_\mu \approx Y_\pi$$

$$y_{22} = Y_o + Y_\mu \approx Y_o$$

$$y_{21} = gm - Y_\mu \approx g_m$$

$$y_{11} = Y_\mu \approx 0$$

Using the formula derived for $G_{TU,max}$ we have

$$G_{TU,max} = \frac{4g_m^2}{\Re(y_{11})\Re(y_{22})}$$

For an ideal FET, the input admittance is imaginary, e.g. $\Re(y_{11}) = 0$, which implies infinite power gain. This is a non-physical result and so we can see that a real FET must have physical resistance on the input side. In practice the gate resistance comes from the poly-gate structure, the interconnect, and the induced channel resistance.

2.3 Two-Port Stability and Negative Resistance

A two-port network is unstable if it supports non-zero currents/voltages with passive terminations

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Since $i_1 = -v_1 Y_S$ and $i_2 = -v_2 Y_L$

$$\begin{pmatrix} y_{11} + Y_S & y_{12} \\ y_{21} & y_{22} + Y_L \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

The only way to have a non-trial solution is for the determinant of the matrix to be zero at a particular frequency. Taking the determinant of the matrix we have

$$(Y_S + y_{11})(Y_L + y_{22}) - y_{12}y_{21} = 0$$

Let's re-write the above in the following form

$$Y_S + y_{11} - \frac{y_{12}y_{21}}{y_{22} + Y_L} = 0$$

or

$$Y_S + Y_{in} = 0$$

equivalently

$$Y_L + Y_{out} = 0$$

A network is unstable at a particular frequency if $Y_S + Y_{in} = 0$, which means the condition is satisfied for both the real and imaginary part. In particular

$$\Re(Y_S + Y_{in}) = \Re(Y_S) + \Re(Y_{in}) = 0$$

Since the terminations are passive, $\Re(Y_S) > 0$ which implies that

$$\Re(Y_{in}) < 0$$

The same equations also show that

$$\Re(Y_{out}) < 0$$

So if these conditions are satisfied, the two-port is unstable.

The conditions for stability are a function of the source and load termination

$$\Re(Y_{in}) = \Re\left(y_{11} - \frac{y_{12}y_{21}}{Y_L + y_{22}}\right) > 0$$

$$\Re(Y_{out}) = \Re\left(y_{22} - \frac{y_{12}y_{21}}{Y_S + y_{11}}\right) > 0$$

For a unilateral amplifier, the conditions are simple and only depend on the two-port

$$\Re(y_{11}) > 0$$

$$\Re(y_{22}) > 0$$

Stability Factor

In general, it can be shown that a two-port is absolutely stable if

$$\Re(y_{11}) > 0$$

$$\Re(y_{22}) > 0$$

and

$$K > 1$$

Where the stability factor K is given by

$$K = \frac{2\Re(y_{11})\Re(y_{22}) - \Re(y_{12}y_{21})}{|y_{12}y_{21}|}$$

The stability of a unilateral amplifier with $y_{12} = 0$ is infinite ($K = \infty$) which implies absolute stability (as long as $\Re(y_{11}) > 0$ and $\Re(y_{22}) > 0$). An amplifier with absolute stability means that the two-port is stable for all passive terminations at either the load or the source. This is a conservative situation in applications where the source and load impedances are well specified and well controlled. But in certain situations the load or source impedance may vary greatly. For instance the input impedance of an antenna can vary if the antenna is moved in proximity to conductors, bent, shorted, or broken. An unstable two-port can be stabilized by adding sufficient loss at the input or output to overcome the negative conductance.

3 Scattering Parameters

Voltages and currents are difficult to measure directly at microwave frequencies. The Z matrix requires "opens", and it's hard to create an ideal open circuit due to parasitic capacitance and radiation. Likewise, a Y matrix requires "shorts", again ideal short circuits are impossible at high frequency due to the finite inductance. Furthermore, many active devices could oscillate under the open or short termination. In practice, we measure scattering or S-parameters at high frequency. The measurement is direct and only involves measurement of relative quantities (such as the standing wave ratio). It's important to realize that although we associate S parameters with high frequency and wave propagation, the concept is valid for any frequency.

3.1 Power Flow in an One-Port

The concept of scattering parameters is very closely related to the concept of power flow. For this reason, we begin with the simple observation that the power flow into a one-port circuit can be written in the following form

$$P_{in} = P_{avs} - P_r$$

where P_{avs} is the available power from the source. Unless otherwise stated, let us assume sinusoidal steady-state. If the source has a real resistance of Z_0, this is simply given by

$$P_{avs} = \frac{V_s^2}{8Z_0}$$

Of course if the one-port is conjugately matched to the source, then it will draw the maximal available power from the source. Otherwise, the power P_{in} is always less than P_{avs}, which is reflected in our equation. In general, P_r represents the wasted or untapped power that one-port circuit is "reflecting" back to the source due to a mismatch. For passive circuits it's clear that each term in the equation is positive and $P_{in} \geq 0$.

The complex power absorbed by the one-port is given by

$$P_{in} = \frac{1}{2}(V_1 \cdot I_1^* + V_1^* \cdot I_1)$$

which allows us to write

$$P_r = P_{avs} - P_{in} = \frac{V_s^2}{4Z_0} - \frac{1}{2}(V_1 I_1^* + V_1^* I_1)$$

the factor of 4 instead of 8 is used since we are now dealing with complex power. The average power can be obtained by taking one half of the real component of the complex power. If the one-port has an input impedance of Z_{in}, then the power P_{in} is expanded to

$$P_{in} = \frac{1}{2}\left(\frac{Z_{in}}{Z_{in}+Z_0}V_s \cdot \frac{V_s^*}{(Z_{in}+Z_0)^*} + \frac{Z_{in}^*}{(Z_{in}+Z_0)^*}V_s^* \cdot \frac{V_s}{(Z_{in}+Z_0)}\right)$$

which is easily simplified to

$$P_{in} = \frac{|V_s|^2}{2Z_0}\left(\frac{Z_0 Z_{in} + Z_{in}^* Z_0}{|Z_{in}+Z_0|^2}\right)$$

where we have assumed Z_0 is real. With the exception of a factor of 2, the premultiplier is simply the source available power, which means that our overall expression for the reflected power is given by

$$P_r = \frac{V_s^2}{4Z_0}\left(1 - 2\frac{Z_0 Z_{in} + Z_{in}^* Z_0}{|Z_{in}+Z_0|^2}\right)$$

which can be simplified

$$P_r = P_{avs}\left|\frac{Z_{in}-Z_0}{Z_{in}+Z_0}\right|^2 = P_{avs}|\Gamma|^2$$

where we have defined Γ, or the reflection coefficient, as

$$\Gamma = \frac{Z_{in}-Z_0}{Z_{in}+Z_0}$$

From the definition it is clear that $|\Gamma| \leq 1$, which is just a re-statement of the conservation of energy implied by our assumption of a passive load. This constant Γ, also called the scattering parameter of a one-port, plays a very important role. On one hand we see that it is has a one-to-one relationship with Z_{in}. Given Γ we can solve for Z_{in} by inverting the above equation

$$Z_{in} = Z_0\frac{1+\Gamma}{1-\Gamma}$$

which means that all of the information in Z_{in} is also in Γ. Moreover, since $|\Gamma| < 1$, we see that the space of the semi-infinite space of all impedance values with real positive components (the right-half plane) maps into the unit circle. This is a great compression of information which allows us to visualize the entire space of realizable impedance values by simply observing the unit circle. We shall find wide application for this concept when finding the appropriate load/source impedance for an amplifier to meet a given noise or gain specification.

More importantly, Γ expresses very direct and obviously the power flow in the circuit. If $\Gamma = 0$, then the one-port is absorbing all the possible power available from the source. If $|\Gamma| = 1$ then the one-port is not absorbing any power, but rather "reflecting" the power back to the source. Clearly an open circuit, short circuit, or a reactive load cannot absorb net power. For an open and short load, this is obvious from the definition of Γ. For a reactive load, this is pretty clear if we substitute $Z_{in} = jX$

$$|\Gamma_X| = \left|\frac{jX - Z_0}{jX + Z_0}\right| = \left|\frac{\sqrt{X^2 + Z_0^2}}{\sqrt{X^2 + Z_0^2}}\right| = 1$$

The transformation between impedance and Γ is a well known mathematical transform (see Bilinear Transform). It is a conformal mapping (meaning that it preserves angles) which maps vertical and horizontal lines in the impedance plane into circles. We have already seen that the jX axis is mapped onto the unit circle.

Since $|\Gamma|^2$ represents power flow, we may imagine that Γ should represent the flow of voltage, current, or some linear combination thereof. Consider taking the square root of the basic equation we have derived

$$\sqrt{P_r} = \Gamma\sqrt{P_{avs}}$$

where we have retained the positive root. We may write the above equation as

$$b_1 = \Gamma a_1$$

where a and b have the units of square root of power and represent signal flow in the network. How are a and b related to currents and voltage? Let

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}}$$

and

$$b_1 = \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}}$$

It is now easy to show that for the one-port circuit, these relations indeed represent the available and reflected power:

$$|a_1|^2 = \frac{|V_1|^2}{4Z_0} + \frac{Z_0|I_1|^2}{4} + \frac{V_1^* \cdot I_1 + V_1 \cdot I_1^*}{4}$$

Now substitute $V_1 = Z_{in}V_s/(Z_{in} + Z_0)$ and $I_1 = V_s/(Z_{in} + Z_0)$ we have

$$|a_1|^2 = \frac{|V_s|^2}{4Z_0}\frac{|Z_{in}|^2}{|Z_{in} + Z_0|^2} + \frac{Z_0|V_s|^2}{4|Z_{in} + Z_0|^2} + \frac{|V_s|^2}{4Z_0}\frac{Z_{in}^* Z_0 + Z_{in}Z_0}{|Z_{in} + Z_0|^2}$$

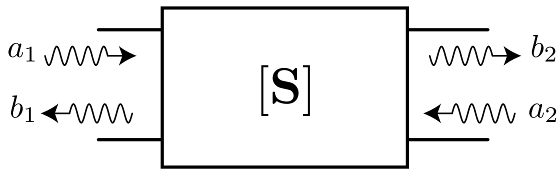


Figure 10: A two-port black box with normalized waves a and b.

or

$$|a_1|^2 = \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}|^2 + Z_0^2 + Z_{in}^* Z_0 + Z_{in}Z_0}{|Z_{in}+Z_0|^2}\right) = \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}+Z_0|^2}{|Z_{in}+Z_0|^2}\right) = P_{avs}$$

In a like manner, the square of b is given by many similar terms

$$|b_1|^2 = \frac{|V_s|^2}{4Z_0}\left(\frac{|Z_{in}|^2 + Z_0^2 - Z_{in}^* Z_0 - Z_{in}Z_0}{|Z_{in}+Z_0|^2}\right) = P_{avs}\left|\frac{Z_{in}-Z_0}{Z_{in}+Z_0}\right|^2 = P_{avs}|\Gamma|^2$$

as expected. We can now see that the expression $b = \Gamma \cdot a$ is analogous to the expression $V = Z \cdot I$ or $I = Y \cdot V$ and so it can be generalized to an N-port circuit. In fact, since a and b are linear combinations of v and i, there is a one-to-one relationship between the two. Taking the sum and difference of a and b we arrive at

$$a_1 + b_1 = \frac{2V_1}{2\sqrt{Z_0}} = \frac{V_1}{\sqrt{Z_0}}$$

which is related to the port voltage and

$$a_1 - b_1 = \frac{2Z_0 I_1}{2\sqrt{Z_0}} = \sqrt{Z_0} I_1$$

which is related to the port current.

3.2 Scattering Parameters for a Two-Port

Let us now generalize the concept of scattering parameters to a two-port and write

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

with reference to Fig. 10, we can interpret the above equation as follows. If we drive a two-port with a source, then a_1 represents the available power from the source, and some fraction of that power will be reflected due to S_{11} (mismatch at the input) and some fraction of that power will "transmitted" to the the second port. In other words, the signal b_2 represents the transmitted signal flowing into the load connected on port two. But if port two is not matched, then this power cannot be fully absorbed and some of that power must flow back into the system, represented by a_2. Let us make this intuitive picture more rigorous by finding the meaning of each parameter. First consider S_{11}, which is easy to to understand if we can set $a_2 = 0$. From the definition of a_2, we have

$$a_2 = \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = 0$$

or

$$\frac{V_2}{-I_2} = Z_0$$

which is tantamount to loading the second port with a resistance of Z_0. Under this condition, then, we can readily identify S_{11}

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$

as simply the same as Γ for a one-port circuit. In other words, this is the ratio of the signal "reflected" back to the source and $1 - |S_{11}|^2$ therefore represents the amount of the available source power flowing into the two-port circuit. Note that this is true as long as the second port is terminated in Z_0. Using the second equation, we have

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

which represents the signal flowing *out* of the two-port and towards the load normalized by the available source power flowing into port 1. In other words, this represents the gain of the two-port under the matched condition. Note that under matched conditions the signals a_1 and b_2 take on particular simply forms

$$a_1 = \frac{V_1 + I_1 Z_0}{2\sqrt{Z_0}} = V\frac{1 + \frac{I_1}{V_1} Z_0}{2\sqrt{Z_0}} = \frac{2V_1}{\sqrt{Z_0}}$$

and

$$b_2 = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}} = V_2\frac{1 - \frac{I_1}{V_1} Z_0}{2\sqrt{Z_0}} = \frac{2V_2}{\sqrt{Z_0}}$$

which means

$$S_{21} = \frac{V_2}{V_1} = 2\frac{V_2}{V_s}$$

which is simply twice the voltage gain of the circuit from the load to the source. This follows since the signal V_1 is exactly half of the source voltage under matched conditions. $|S_{21}|^2$ is the power gain of the two-port when both ports are terminated by Z_0 since in this case all the available source power flows into the two port and the amount appearing at the load is given by $|b_2|^2$. If $|S_{21}| > 1$, that means there is more power at the load than power flowing into the two-port, which can only be true if the two-port is active. If we interchange the order of the ports, we immediately see that S_{22} is likewise the output reflection coefficient under matched conditions and S_{12} is the reverse gain of the two-port.

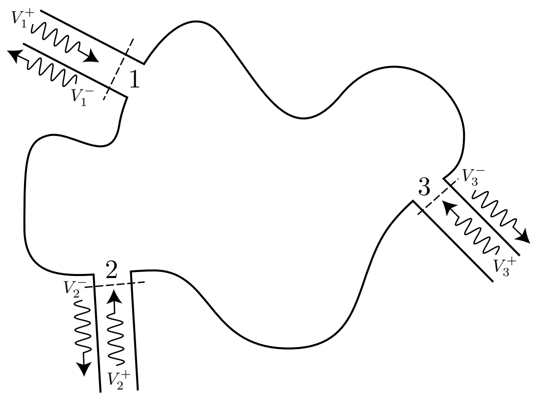


Figure 11: An arbitrary N port circuit with incident and reflected waves at each port.

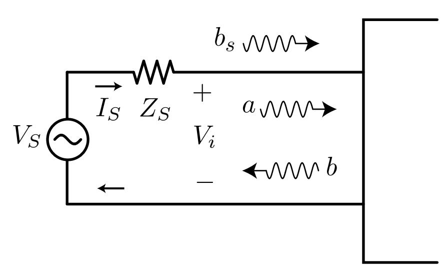


Figure 12: A voltage source with source impedance Z_S.

3.3 Representation of Source

How do we represent the voltage source in Fig. 12 with a source impedance $Z_s \neq Z_0$ directly with S parameters? Start with the I-V relation

$$V_i = V_s - I_s Z_s$$

The voltage source can be represented directly for s-parameter analysis as follows. First note that

$$(a+b)\sqrt{Z_0} = V_s - \left(\frac{a-b}{\sqrt{Z_0}}\right) Z_s$$

or

$$b(Z_0 - Z_s) = \sqrt{Z_0}V_s - a(Z_s + Z_0)$$

Solve these equations for a, the power flowing into a two-port

$$a = \frac{\sqrt{Z_0}V_s}{Z_s + Z_0} + b\frac{Z_0 - Z_s}{Z_0 + Z_s}$$

Define Γ_s as the source reflection coefficient and b_s as the source signal

$$\Gamma_s = \frac{Z_0 - Z_s}{Z_0 + Z_s}$$

$$b_s = \frac{\sqrt{Z_0}V_s}{Z_s + Z_0}$$

With these definitions in place, the power flow away from the source has a simple form

$$a = b_s + b\Gamma_s$$

If the source is matched to Z_0, then $\Gamma_s = 0$ and the total power flowing out of the source is the same as the source power. Otherwise the source signal power should include any reflections occurring at the source itself.

Available Power from Source

A useful quantity is the available power from a source under conjugate matched conditions. Let's begin by noting that the power flowing into a load Γ_L is given by

$$P_L = |a|^2 - |b|^2 = |a|^2(1 - |\Gamma_L|^2)$$

Using the fact that $b = \Gamma_L a$, the input power signal is given by

$$a = b_s + b\Gamma_s = b_s + \Gamma_L \Gamma_s a$$

or

$$a = \frac{b_s}{1 - \Gamma_L \Gamma_s}$$

Therefore the power flowing into the load is given by

$$P_L = \frac{|b_s|^2(1-|\Gamma_L|^2)}{|1-\Gamma_L\Gamma_s|^2}$$

To draw the available power from the source, we should conjugately match the load $\Gamma_L = \Gamma_s^*$

$$P_{avs} = P_L\big|_{\Gamma_L=\Gamma_s^*} = \frac{|b_s|^2(1-|\Gamma_s|^2)}{|1-|\Gamma_s|^2|^2} = \frac{|b_s|^2}{1-|\Gamma_s|^2}$$

3.4 Incident and Scattering Waves

If you're familiar with transmission line theory, then you clearly understand the origin of the term "reflected" signal and "transmitted" signal. In transmission line theory, signal a is often called the "forward" wave and represetned by v^+ and b is called the reflected or scattered wave and denoted by v^-. In a transmission line the power is actually reflected since the source does not know the port impedance until information travels from the source to the two-port and then back to the source again (limited by the speed of light) and so there is a physical origin to the terminology. In lumped circuit theory, there is no time delay, but we use the same terminology. For an N port circuit, consider N transmission line connected to each port (Fig. 11) and define the reference plane as the point where the transmission line terminates onto the port. In transmission line parlance, these signals are voltages (and currents), so we define them as follows

$$v^+ = V + IZ_0$$

$$v^- = V - IZ_0$$

Notice the similarity to the definition of a and b, where the normalization and power factors are missing. The vectors v^- and v^+ are the incident and "scattered" waveforms

$$v^+ = \begin{pmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \\ \vdots \end{pmatrix}$$

$$v^- = \begin{pmatrix} V_1^- \\ V_2^- \\ V_3^- \\ \vdots \end{pmatrix}$$

Because the N port is linear, we expect that scattered field to be a linear function of the incident field

$$v^- = Sv^+$$

S is the scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots \\ S_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

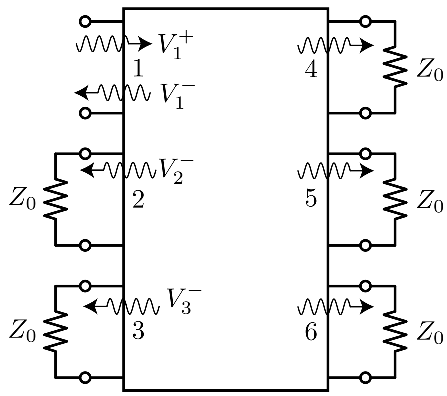


Figure 13: An N port circuit with all ports terminated so that $V_j^+ = 0$ for $j \neq 1$.

The fact that the S matrix exists can be easily proved using transmission line theory. The voltage and current on each transmission line termination can be written as

$$V_i = V_i^+ + V_i^-$$

$$I_i = Y_0(I_i^+ - I_i^-)$$

Inverting these equations

$$V_i + Z_0 I_i = V_i^+ + V_i^- + V_i^+ - V_i^- = 2V_i^+$$

$$V_i - Z_0 I_i = V_i^+ + V_i^- - V_i^+ + V_i^- = 2V_i^-$$

Thus v^+,v^- are simply linear combinations of the port voltages and currents. By the uniqueness theorem, then, $v^- = Sv^+$.

Measurement of S_{ij}

The term S_{ij} can be computed directly by the following formula

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \, \forall \, k \neq j}$$

Solve for $V_k^+ = 0$

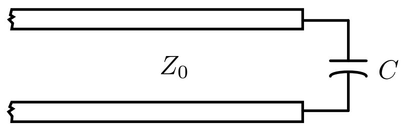
$$V_k^+ = V_k + I_k Z_0 = 0$$

or

$$\frac{V_k}{-I_k} = Z_0$$

which means we terminate port k with an impedance Z_0 and measure the scattered waves. From a transmission line perspective, to measure S_{ij}, drive port j with a wave amplitude of V_j^+ and terminate all other ports with the characteristic impedance of the lines (so that $V_k^+ = 0$ for $k \neq j$), as shown in Fig. 13. Then observe the wave amplitude coming out of the port i.

Example 1:



Let's calculate the S parameter for a capacitor

$$S_{11} = \frac{V_1^-}{V_1^+}$$

We can also do the calculation directly from the definition of S parameters. Substituting for the current in a capacitor

$$V_1^- = V - IZ_0 = V - j\omega CV = V(1 - j\omega CZ_0)$$

$$V_1^+ = V + IZ_0 = V + j\omega CV = V(1 + j\omega CZ_0)$$

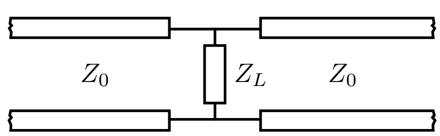
Alternatively, this is just the reflection coefficient for a capacitor

$$S_{11} = \rho_L = \frac{Z_C - Z_0}{Z_C + Z_0} = \frac{\frac{1}{j\omega C} - Z_0}{\frac{1}{j\omega C} + Z_0}$$

$$= \frac{1 - j\omega CZ_0}{1 + j\omega CZ_0}$$

and the ratio yields the same result as expected.

Example 2:



Consider a shunt impedance connected at the junction of two transmission lines. If we terminate port 2 in an impedance Z_0, then the current $I_1 = V_1/R||Z_0$, which allows us to write

$$V_1^- = V_1 - I_1 Z_0 = V_1 \left(1 - \frac{Z_0}{R||Z_0}\right)$$

In a like manner, the incident wave is given by

$$V_1^+ = V_1 + I_1 Z_0 = V_1 \left(1 + \frac{Z_0}{R||Z_0}\right)$$

The ratio gives us the scattering coefficient

$$S_{11} = \frac{1 - \frac{Z_0}{R||Z_0}}{1 + \frac{Z_0}{R||Z_0}} = \frac{R||Z_0 - Z_0}{R||Z_0 + Z_0}$$

From transmission line theory, we recognize this to be the reflection coefficient seen at port one when port two is terminated in Z_0. We can also calculate S_{21} by noting that

$$V_2^- = V_2 - Z_0 I_2 = V_1 - Z_0 \left(\frac{-V_1}{Z_0}\right) = 2V_1$$

Taking the ratio with the incident wave V_1^+

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^-=0} = \frac{2}{1 + \frac{Z_0}{R||Z_0}} = \frac{2R||Z_0}{R||Z_0 + Z_0}$$

By symmetry, we have the complete two-port scattering parameters. Another approach is to use transmission line theory. Start by observing that the voltage at the junction is continuous. The currents, though, differ

$$V_1 = V_2$$

$$I_1 + I_2 = Y_L V_2$$

To compute S_{11}, enforce $V_2^+ = 0$ by terminating the line. Thus we can be re-write the above equations

$$V_1^+ + V_1^- = V_2^-$$

$$Y_0(V_1^+ - V_1^-) = Y_0 V_2^- + Y_L V_2^- = (Y_L + Y_0)V_2^-$$

We can now solve the above equation for the reflected and transmitted wave

$$V_1^- = V_2^- - V_1^+ = \frac{Y_0}{Y_L + Y_0}(V_1^+ - V_1^-) - V_1^+$$

$$V_1^-(Y_L + Y_0 + Y_0) = (Y_0 - (Y_0 + Y_L))V_1^+$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{Y_0 - (Y_0 + Y_L)}{Y_0 + (Y_L + Y_0)} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

The above equation can be written by inspection since $Z_0||Z_L$ is the effective load seen at the junction of port 1. Thus for port 2 we can write

$$S_{22} = \frac{Z_0||Z_L - Z_0}{Z_0||Z_L + Z_0}$$

Likewise, we can solve for the transmitted wave, or the wave scattered into port 2

$$S_{21} = \frac{V_2^-}{V_1^+}$$

Since $V_2^- = V_1^+ + V_1^-$, we have

$$S_{21} = 1 + S_{11} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

By symmetry, we can deduce S_{12} as

$$S_{12} = \frac{2Z_0||Z_L}{Z_0||Z_L + Z_0}$$

Conversion Formula

Since V^+ and V^- are related to V and I, it's easy to find a formula to convert for Z or Y to S

$$V_i = V_i^+ + V_i^- \;\rightarrow\; v = v^+ + v^-$$

$$Z_{i0} I_i = V_i^+ - V_i^- \;\rightarrow\; Z_0 i = v^+ - v^-$$

Now starting with $v = Zi$, we have

$$v^+ + v^- = Z Z_0^{-1}(v^+ - v^-)$$

Note that Z_0 is the scalar port impedance

$$v^-(I + Z Z_0^{-1}) = (Z Z_0^{-1} - I)v^+$$

$$v^- = (I + Z Z_0^{-1})^{-1}(Z Z_0^{-1} - I)v^+ = S v^+$$

We now have a formula relating the Z matrix to the S matrix

$$S = (Z Z_0^{-1} + I)^{-1}(Z Z_0^{-1} - I) = (Z + Z_0 I)^{-1}(Z - Z_0 I)$$

Recall that the reflection coefficient for a load is given by the same equation!

$$\overline{\rho} = \frac{Z/Z_0 - 1}{Z/Z_0 + 1}$$

To solve for Z in terms of S, simply invert the relation

$$Z_0^{-1} Z S + I S = Z_0^{-1} Z - I$$

$$Z_0^{-1} Z (I - S) = S + I$$

$$Z = Z_0 (I + S)(I - S)^{-1}$$

As expected, these equations degenerate into the correct form for a 1×1 system

$$Z_{11} = Z_0 \frac{1 + S_{11}}{1 - S_{11}}$$

Reciprocal Networks

We have found that the Z and Y matrix are symmetric. Now let's see what we can infer about the S matrix.

$$v^+ = \frac{1}{2}(v + Z_0 i)$$

$$v^- = \frac{1}{2}(v - Z_0 i)$$

Substitute $v = Zi$ in the above equations

$$v^+ = \frac{1}{2}(Zi + Z_0 i) = \frac{1}{2}(Z + Z_0)i$$

$$v^- = \frac{1}{2}(Zi - Z_0 i) = \frac{1}{2}(Z - Z_0)i$$

Since $i = i$, the above equation must result in consistent values of i

$$2(Z + Z_0)^{-1}v^+ = 2(Z - Z_0)^{-1}v^-$$

Thus

$$S = (Z - Z_0)(Z + Z_0)^{-1}$$

Consider the transpose of the S matrix

$$S^t = \left((Z + Z_0)^{-1}\right)^t (Z - Z_0)^t$$

Recall that Z_0 is a diagonal matrix

$$S^t = (Z^t + Z_0)^{-1}(Z^t - Z_0)$$

If $Z^t = Z$ (reciprocal network), then we have

$$S^t = (Z + Z_0)^{-1}(Z - Z_0)$$

Previously we found that

$$S = (Z + Z_0)^{-1}(Z - Z_0)$$

So that we see that the S matrix is also symmetric (under reciprocity)

$$S^t = S$$

To see this another way, note that in effect we have shown that

$$(Z + I)^{-1}(Z - I) = (Z - I)(Z + I)^{-1}$$

This is easy to demonstrate if we note that

$$Z^2 - I = Z^2 - I^2 = (Z + I)(Z - I) = (Z - I)(Z + I)$$

In general matrix multiplication does not commute, but here it does

$$(Z - I) = (Z + I)(Z - I)(Z + I)^{-1}$$

$$(Z + I)^{-1}(Z - I) = (Z - I)(Z + I)^{-1}$$

Thus we see that $S^t = S$.

Scattering Parameters of a Lossless Network

Consider the total power dissipated by a lossless network (must sum to zero)

$$P_{av} = \frac{1}{2}\Re\left(v^t i^*\right) = 0$$

Expanding in terms of the wave amplitudes

$$= \frac{1}{2}\Re\left((v^+ + v^-)^t Z_0^{-1}(v^+ - v^-)^*\right)$$

where we assume that Z_0 are real numbers and equal. The notation is about to get ugly in the expansion

$$= \frac{1}{2Z_0}\Re\left(v^{+t}v^{+*} - v^{+t}v^{-*} + v^{-t}v^{+*} - v^{-t}v^{-*}\right)$$

The middle terms sum to a purely imaginary number. Let $x = v^+$ and $y = v^-$

$$y^t x^* - x^t y^* = y_1 x_1^* + y_2 x_2^* + \cdots - x_1 y_1^* + x_2 y_2^* + \cdots = a - a^*$$

We have shown that

$$P_{av} = \frac{1}{2Z_0}\left(\underbrace{v^{+t}v^{+}}_{\text{total incident power}} - \underbrace{v^{-t}v^{-*}}_{\text{total reflected power}}\right) = 0$$

This is a rather obvious result. It simply says that the incident power is equal to the reflected power (because the N port is lossless). Since $v^- = Sv^+$

$$v^{+t}v^{+} = (Sv^+)^t(Sv^+)^* = v^{+t}S^tS^*v^{+*}$$

This can only be true if S is a unitary matrix

$$S^tS^* = I$$

$$S^* = (S^t)^{-1}$$

Orthogonal Properties of S

Expanding out the matrix product

$$\delta_{ij} = \sum_k (S^T)_{ik}S^*_{kj} = \sum_k S_{ki}S^*_{kj}$$

For $i = j$ we have

$$\sum_k S_{ki}S^*_{ki} = 1$$

For $i \neq j$ we have

$$\sum_k S_{ki}S^*_{kj} = 0$$

The dot product of any column of S with the conjugate of that column is unity while the dot product of any column with the conjugate of a different column is zero. If the network is reciprocal, then $S^t = S$ and the same applies to the rows of S. Note also that $|S_{ij}| \leq 1$.

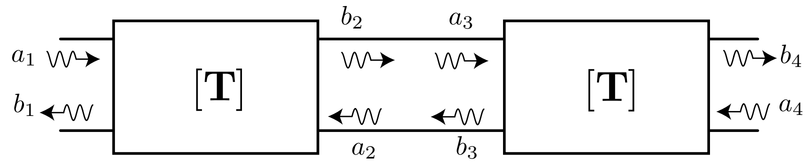


Figure 14: The cascade of two two-ports. The incident and reflected power at the connection point.

Shift in Reference Planes

A convenient feature of the scattering parameters is that we can easily move the reference plane. In other words, if we connect transmission lines of arbitrary length to any port, we can easily de-embed their effect. We'll derive a new matrix S' related to S. Let's call the waves at the new reference ν

$$v^- = Sv^+$$

$$\nu^- = S'\nu^+$$

Since the waves on the lossless transmission lines only experience a phase shift, we have a phase shift of $\theta_i = \beta_i \ell_i$

$$\nu_i^- = v^- e^{-j\theta_i}$$

$$\nu_i^+ = v^+ e^{j\theta_i}$$

Or we have

$$\begin{bmatrix} e^{j\theta_1} & 0 & \cdots & \\ 0 & e^{j\theta_2} & \cdots & \\ 0 & 0 & e^{j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} \nu^- = S \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} \nu^+$$

So we see that the new S matrix is simply

$$S' = \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix} S \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & \\ 0 & e^{-j\theta_2} & \cdots & \\ 0 & 0 & e^{-j\theta_3} & \cdots \\ \vdots & & & \end{bmatrix}$$

3.5 Scattering Transfer Parameters

Up to now we found it convenient to represent the scattered waves in terms of the incident waves. But what if we wish to cascade two ports as shown in Fig. 14? Since b_2 flows into a_1', and likewise b_1' flows into a_2, would it not be convenient if we defined the a relationship between a_1,b_1 and b_2,a_2? In other words we have

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

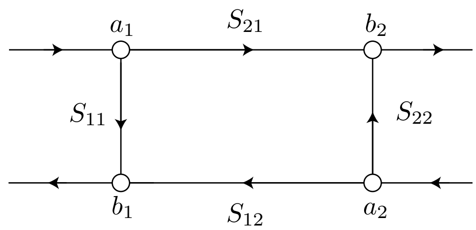


Figure 15: The signal-flow graph of a two-port.

Notice carefully the order of waves (a,b) in reference to the figure above. This allows us to cascade matrices

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = T_1 \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = T_1 \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = T_1 T_2 \begin{bmatrix} b_4 \\ a_4 \end{bmatrix}$$

4 Signal-Flow Analysis

Signal-flow analysis is a technique for graphically calculating the transfer function directly using scattering parameters. Each signal a and b in the system is represented by a node. Branches connect nodes with "strength" given by the scattering parameter. For example, a general two-port is represented in Fig. 15. Using three simple rules, we can simplify signal flow graphs to the point that detailed calculations are done by inspection. Of course we can always "do the math" using algebra, so pick the technique that you like best.

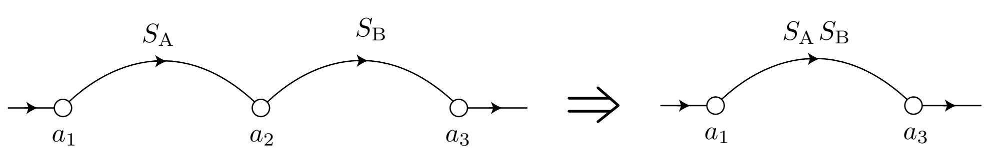


Figure 16: The series connection rule.

- Rule 1: (series rule) By inspection of Fig. 16, we have the cascade.

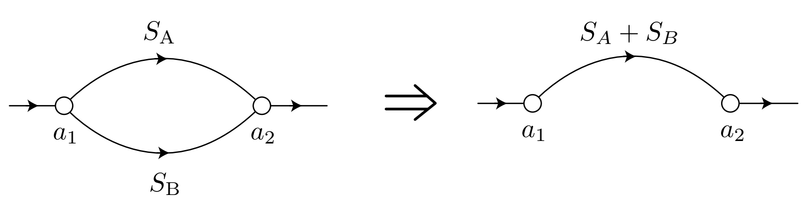


Figure 17: The parallel connection rule.

- Rule 2: (parallel rule) Clear by inspection of Fig. 17.

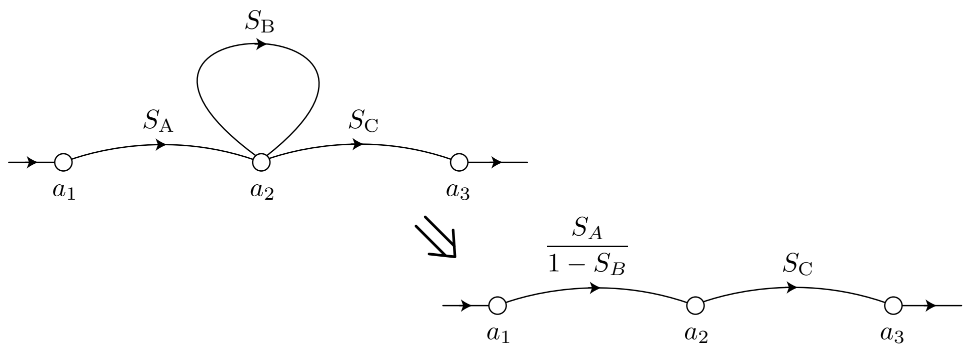


Figure 18: The self-loop elimination rule.

- Rule 3: (self-loop rule) We can remove a "self-loop" in Fig. 18 by multiplying branches feeding the node by $1/(1-S_B)$ since

$$a_2 = S_A a_1 + S_B a_2$$
$$a_2(1-S_B) = S_A a_1$$
$$a_2 = \frac{S_A}{1-S_B} a_1$$

- Rule 4: (splitting rule) We can duplicate node a_2 in Fig. 19 by splitting the signals at an earlier phase

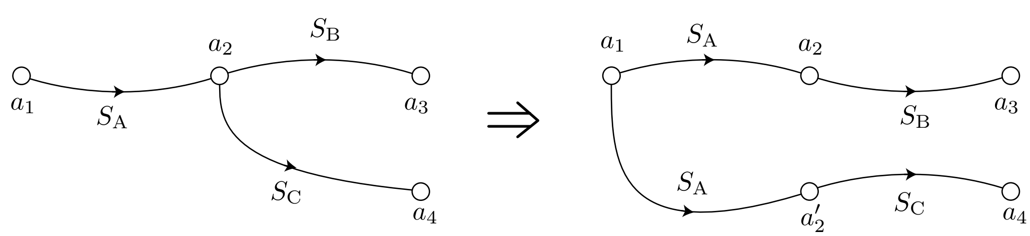


Figure 19: The splitting rule.

Using the above rules, we can calculate the input reflection coefficient of a two-port terminated by $\Gamma_L = b_1/a_1$ shown in Fig. 20a using a couple of steps. First we notice that there is a self-loop around b_2 (Fig. 20b). Next we remove the self loop and from here it's clear that the (Fig. 20c)

$$\Gamma_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1-S_{22}\Gamma_L}$$

4.1 Mason's Rule

Using Mason's Rule, you can calculate the transfer function for a signal flow graph by "inspection"

$$T = \frac{P_1\left(1-\sum\mathcal{L}(1)^{(1)}+\sum\mathcal{L}(2)^{(1)}-\cdots\right)+P_2\left(1-\sum\mathcal{L}(1)^{(2)}+\cdots\right)+\cdots}{1-\sum\mathcal{L}(1)+\sum\mathcal{L}(2)-\sum\mathcal{L}(3)+\cdots}$$

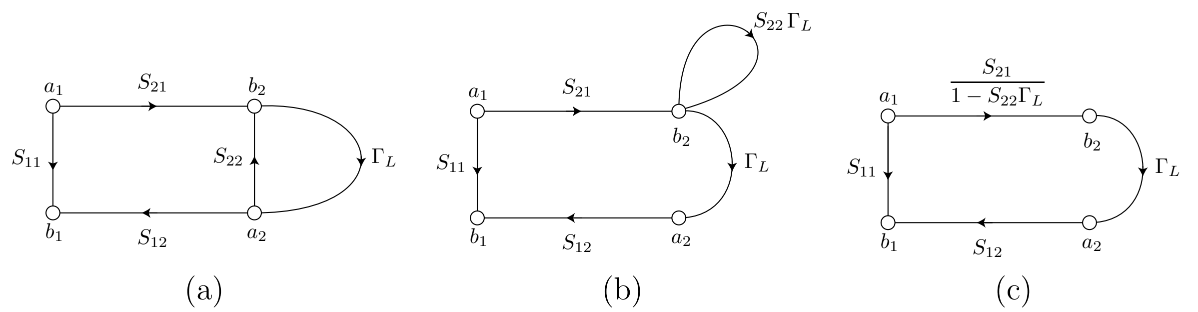


Figure 20: (a) A two-port terminated in a load Γ_L. (b) Identification of the self-loop. (c) Elimination of the self-loop.

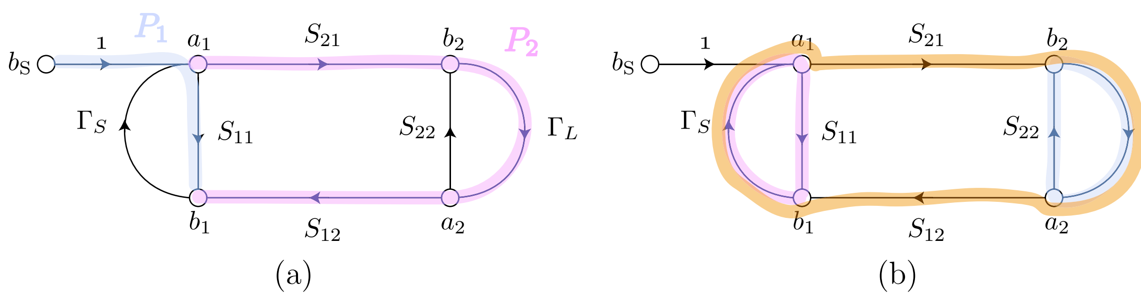


Figure 21: (a) Identification of the paths in a signal-flow graph. (b) Identification of the loops in a signal-flow graph.

Each P_i defines a *path*, a directed route from the input to the output not containing each node more than once. The value of P_i is the product of the branch coefficients along the path. For instance, in Fig. 21a, the path from b_s to b_1 ($T = b_1/b_s$) has two paths, $P_1 = S_{11}$ and $P_2 = S_{21}\Gamma_L S_{12}$

Loop of Order Summation Notation

The notation $\sum \mathcal{L}(1)$ is the sum over all first order loops. A "first order loop" is defined as product of the branch values in a loop in the graph. For the example shown in Fig. 21b, we have $\Gamma_s S_{11}$, $S_{22}\Gamma_L$, and $\Gamma_s S_{21}\Gamma_L S_{12}$. A "second order loop" $\mathcal{L}(2)$ is the product of two non-touching first-order loops. For instance, since loops $S_{11}\Gamma_s$ and $S_{22}\Gamma_L$ do not touch, their product is a second order loop. A "third order loop" $\mathcal{L}(3)$ is likewise the product of three non-touching first order loops. The notation $\sum \mathcal{L}(1)^{(p)}$ is the sum of all first-order loops that do not touch the path p. For path P_1, we have $\sum \mathcal{L}(1)^{(1)} = \Gamma_L S_{22}$ but for path P_2 we have $\sum \mathcal{L}(1)^{(2)} = 0$.

Example 3:

Input Reflection of Two-Port

Let's redo the calculation of $\Gamma_{in} = b_1/a_1$ for the signal-flow graph shown in Fig. 20. Using Mason's rule, you can quickly identify the relevant paths. There are two paths $P_1 = S_{11}$ and $P_2 = S_{21}\Gamma_L S_{12}$. There is only one first-order loop: $\sum \mathcal{L}(1) = S_{22}\Gamma_L$ and so naturally there are no higher order loops. Note that the loop does not touch path P_1, so $\sum \mathcal{L}(1)^{(1)} = S_{22}\Gamma_L$. Now let's apply Mason's general formula

$$\Gamma_{in} = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{21}\Gamma_L S_{12}}{1 - S_{22}\Gamma_L} = S_{11} + \frac{S_{21}\Gamma_L S_{12}}{1 - S_{22}\Gamma_L}$$

Example 4:

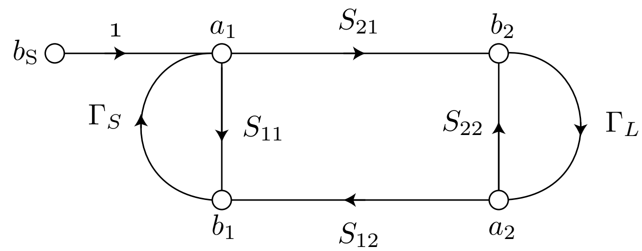


Figure 22: A two-port driven by a source with reflection coefficient Γ_S and loaded by Γ_L.

Transducer Power Gain

By definition, the transducer power gain for the two-port shown in Fig. 22 is given by

$$G_T = \frac{P_L}{P_{AVS}} = \frac{|b_2|^2(1 - |\Gamma_L|^2)}{\frac{|b_s|^2}{1-|\Gamma_S|^2}}$$

$$= \left|\frac{b_2}{b_S}\right|^2 (1 - |\Gamma_L|^2)(1 - |\Gamma_S|^2)$$

By Mason's Rule, there is only one path $P_1 = S_{21}$ from b_S to b_2 so we have

$$\sum \mathcal{L}(1) = \Gamma_S S_{11} + S_{22}\Gamma_L + \Gamma_S S_{21}\Gamma_L S_{12}$$

$$\sum \mathcal{L}(2) = \Gamma_S S_{11}\Gamma_L S_{22}$$

$$\sum \mathcal{L}(1)^{(1)} = 0$$

The gain expression is thus given by

$$\frac{b_2}{b_S} = \frac{S_{21}(1-0)}{1 - \Gamma_S S_{11} - S_{22}\Gamma_L - \Gamma_S S_{21}\Gamma_L S_{12} + \Gamma_S S_{11}\Gamma_L S_{22}}$$

The denominator is in the form of $1 - x - y + xy$ which allows us to write

$$G_T = \frac{|S_{21}|^2(1-|\Gamma_S|^2)(1-|\Gamma_L|^2)}{|(1-S_{11}\Gamma_S)(1-S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_L\Gamma_S|^2}$$

Recall that $\Gamma_{in} = S_{11} + S_{21}S_{12}\Gamma_L/(1-S_{22}\Gamma_L)$. Factoring out $1 - S_{22}\Gamma_L$ from the denominator we have

$$\text{den} = \left(1 - S_{11}\Gamma_S - \frac{S_{21}S_{12}\Gamma_L}{1-S_{22}\Gamma_L}\Gamma_S\right)(1-S_{22}\Gamma_L)$$

$$\begin{aligned}\text{den} &= \left(1 - \Gamma_S\left(S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1-S_{22}\Gamma_L}\right)\right)(1-S_{22}\Gamma_L)\\ &= (1-\Gamma_S\Gamma_{in})(1-S_{22}\Gamma_L)\end{aligned}$$

This simplifications allows us to write the transducer gain in the following convenient form

$$G_T = \frac{1-|\Gamma_S|^2}{|1-\Gamma_{in}\Gamma_S|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}$$

which can be viewed as a product of the action of the input match "gain", the intrinsic two-port gain $|S_{21}|^2$, and the output match "gain". Since the general two-port is not unilateral, the input match is a function of the load. Likewise, by symmetry we can also factor the expression to obtain

$$G_T = \frac{1-|\Gamma_S|^2}{|1-S_{11}\Gamma_S|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\Gamma_L|^2}$$

---

5 Stability of a Two-Port

A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condition, the two-port can oscillate. Consider the input admittance

$$Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22}+Y_L} \tag{2}$$

Using the following definitions

$$Y_{11} = g_{11} + jb_{11} \tag{3}$$

$$Y_{22} = g_{22} + jb_{22} \tag{4}$$

$$Y_{12}Y_{21} = P + jQ = L\angle\phi \tag{5}$$

$$Y_L = G_L + jB_L \tag{6}$$

Now substitute real/imaginary parts of the above quantities into Y_{in}

$$Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L} \tag{7}$$

$$= g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2} \tag{8}$$

Taking the real part, we have the input conductance

$$\Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22} + G_L) + Q(b_{22} + B_L)}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2} \tag{9}$$

$$= \frac{(g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L)}{D} \tag{10}$$

Since $D > 0$ if $g_{11} > 0$, we can focus on the numerator. Note that $g_{11} > 0$ is a requirement since otherwise oscillations would occur for a short circuit at port 2. The numerator can be factored into several positive terms

$$N = (g_{22} + G_L)^2 + (b_{22} + B_L)^2 - \frac{P}{g_{11}}(g_{22} + G_L) - \frac{Q}{g_{11}}(b_{22} + B_L) \tag{11}$$

$$= \left(G_L + \left(g_{22} - \frac{P}{2g_{11}}\right)\right)^2 + \left(B_L + \left(b_{22} - \frac{Q}{2g_{11}}\right)\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} \tag{12}$$

Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left(b_{22} - \frac{Q}{2g_{11}}\right)$ (reactive load)

$$N_{min} = \left(g_{22} - \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} \tag{13}$$

And thus the above must remain positive, $N_{min} > 0$, so

$$\left(g_{22} - \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0 \tag{14}$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos\phi) \tag{15}$$

5.0.1 Linvill/Llewellyn Stability Factors

Using the above equation, we define the Linvill stability factor

$$L < 2g_{11}g_{22} - P \tag{16}$$

$$C = \frac{L}{2g_{11}g_{22} - P} < 1 \tag{17}$$

The two-port is stable if $0 < C < 1$. It's more common to use the inverse of C as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1 \tag{18}$$

The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1 \tag{19}$$

The above expression is identical if we interchange ports 1/2. Thus it's the general condition for stability. Note that $K > 1$ is the same condition for the maximum stable gain derived last section. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!

5.1 Stability from Scattering Parameters

We can also derive stability in terms of the input reflection coefficient. For a general two-port with load Γ_L we have

$$v_2^- = \Gamma_L^{-1} v_2^+ = S_{21}v_1^+ + S_{22}v_2^+ \tag{20}$$

$$v_2^+ = \frac{S_{21}}{\Gamma_L^{-1} - S_{22}} v_1^- \tag{21}$$

$$v_1^- = \left(S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - \Gamma_L S_{22}} \right) v_1^+ \tag{22}$$

$$\Gamma = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - \Gamma_L S_{22}} \tag{23}$$

If $|\Gamma| < 1$ for all Γ_L, then the two-port is stable

$$\Gamma = \frac{S_{11}(1 - S_{22}\Gamma_L) + S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} + \Gamma_L(S_{21}S_{12} - S_{11}S_{22})}{1 - S_{22}\Gamma_L} \tag{24}$$

$$= \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \tag{25}$$

To find the boundary between stability/instability, let's set $|\Gamma| = 1$

$$\left| \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \right| = 1 \tag{26}$$

$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L| \tag{27}$$

After some algebraic manipulations, we arrive at the following equation

$$\left|\Gamma - \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2}\right| = \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} \tag{28}$$

This is of course the equation of a circle, $|\Gamma - C| = R$, in the complex plane with center at C and radius R. Thus a circle on the Smith Chart divides the region of instability from stability.

Consider the stability circle for a unilateral two-port

$$C_S = \frac{S_{11}^* - (S_{11}^* S_{22}^*)S_{22}}{|S_{11}|^2 - |S_{11}S_{22}|^2} = \frac{S_{11}^*}{|S_{11}|^2} \tag{29}$$

$$R_S = 0 \tag{30}$$

$$|C_S| = \frac{1}{|S_{11}|} \tag{31}$$

The center of the circle lies outside of the unit circle if $|S_{11}| < 1$. The same is true of the load stability circle. Since the radius is zero, stability is only determined by the location of the center. If $S_{12} = 0$, then the two-port is unconditionally stable if $S_{11} < 1$ and $S_{22} < 1$. This result is trivial since

$$\Gamma_S\,|_{S_{12}=0} = S_{11} \tag{32}$$

The stability of the source depends only on the device and not on the load.

5.2 μ Stability Test

If we want to determine if a two-port is unconditionally stable, then we should use the μ-test

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1 \tag{33}$$

The μ-test not only is a test for unconditional stability, but the magnitude of μ is a measure of the stability. In other words, if one two-port has a larger μ, it is more stable.

The advantage of the μ-test is that only a single parameter needs to be evaluated. There are no auxiliary conditions like the K-test derivation earlier. The derivation of the μ-test can proceed as follows. First let $\Gamma_S = |\rho_s|e^{j\phi}$ and evaluate Γ_{out}

$$\Gamma_{out} = \frac{S_{22} - \Delta|\rho_s|e^{j\phi}}{1 - S_{11}|\rho_s|e^{j\phi}} \tag{34}$$

Next we can manipulate this equation into the equation for a circle $|\Gamma_{out} - C| = R$

$$\left|\Gamma_{out} + \frac{|\rho_s|S_{11}^*\Delta - S_{22}}{1 - |\rho_s||S_{11}|^2}\right| = \frac{\sqrt{|\rho_s|}|S_{12}S_{21}|}{(1 - |\rho_s||S_{11}|^2)} \tag{35}$$

For a two-port to be unconditionally stable, we'd like Γ_{out} to fall within the unit circle

$$||C| + R| < 1 \tag{36}$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| < 1 - |\rho_s||S_{11}|^2 \tag{37}$$

$$||\rho_s|S_{11}^*\Delta - S_{22}| + \sqrt{|\rho_s|}|S_{21}S_{12}| + |\rho_s||S_{11}|^2 < 1 \tag{38}$$

The worst case stability occurs when $|\rho_s| = 1$ since it maximizes the left-hand side of the equation. Therefore we have

$$\mu = \frac{1 - |S_{11}|^2}{|S_{11}^*\Delta - S_{22}| + |S_{12}S_{21}|} > 1 \tag{39}$$

5.2.1 K-Δ Test

The K stability test has already been derived using Y parameters. We can also do a derivation based on S parameters. This form of the equation has been attributed to Rollett and Kurokawa. The idea is very simple and similar to the μ test. We simply require that all points in the instability region fall outside of the unit circle. The stability circle will intersect with the unit circle if

$$|C_L| - R_L > 1 \tag{40}$$

or

$$\frac{|S_{22}^* - \Delta^* S_{11}| - |S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} > 1 \tag{41}$$

This can be recast into the following form (assuming $|\Delta| < 1$)

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}||S_{21}|} > 1 \tag{42}$$

5.3 N-Port Passivity

We would like to find if an N-port is active or passive. Passivity is different from stability, and plays an important role in determining the maximum frequency of operation for an "active" device. For instance, above a certain frequency every transistor will transition from an active device to a passive device, setting an upper limit for amplification or oscillation with a given device. By definition, an N-port is passive if it can only absorb net power. The total net complex power flowing into or out of a N port is given by

$$P = (V_1^* I_1 + V_2^* I_2 + \cdots) = (I_1^* V_1 + I_2^* V_2 + \cdots) \tag{43}$$

If we sum the above two terms we have

$$P = \frac{1}{2}(v^*)^T i + \frac{1}{2}(i^*)^T v \tag{44}$$

for vectors of current and voltage i and v. Using the admittance matrix $i = Yv$, this can be recast as

$$P = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(Y^* v^*)^T v = \frac{1}{2}(v^*)^T Y v + \frac{1}{2}(v^*)^T (Y^*)^T v \tag{45}$$

$$P = (v^*)^T \frac{1}{2}(Y + (Y^*)^T) v = (v^*)^T Y_H v \tag{46}$$

Thus for a network to be passive, the Hermitian part of the matrix Y_H should be positive semi-definite.

For a two-port, the condition for passivity can be simplified as follows. Let the general hybrid admittance matrix for the two-port be given by

$$H(s) = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + j \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \tag{47}$$

$$H_H(s) = \frac{1}{2}(H(s) + H^*(s)) \tag{48}$$

$$= \begin{pmatrix} m_{11} & \frac{1}{2}((m_{12} + m_{21}) + j(n_{12} - n_{21})) \\ ((m_{12} + m_{21}) + j(n_{21} - n_{12})) & m_{22} \end{pmatrix} \tag{49}$$

This matrix is positive semi-definite if

$$m_{11} > 0 \tag{50}$$

$$m_{22} > 0 \tag{51}$$

$$\det H_n(s) \geq 0 \tag{52}$$

or

$$4m_{11}m_{22} - |k_{12}|^2 - |k_{21}|^2 - 2\Re(k_{12}k_{21}) \geq 0 \tag{53}$$

$$4m_{11}m_{22} \geq |k_{12} + k_{21}^*|^2 \tag{54}$$

Example 5:

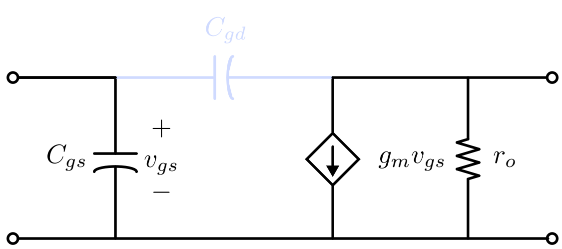


Figure 23: A simplified hybrid-π equivalent circuit.

A simple equivalent circuit for a FET without any feedback, shown in Fig. 23, is of course absolutely stable if the resistors of the model are positive. The Z matrix for the circuit is given by

$$Z = \begin{bmatrix} \frac{1}{j\omega C_{gs}} & 0 \\ \frac{-g_m r_o}{j\omega C_{gs}} & r_o \end{bmatrix} \tag{55}$$

Since $Z_{12} = 0$, the stability factor $K = \infty$

$$K = \frac{2\Re(Z_{11})\Re(Z_{22}) - \Re(Z_{12}Z_{21})}{|Z_{12}Z_{21}|} \tag{56}$$

Example 6:

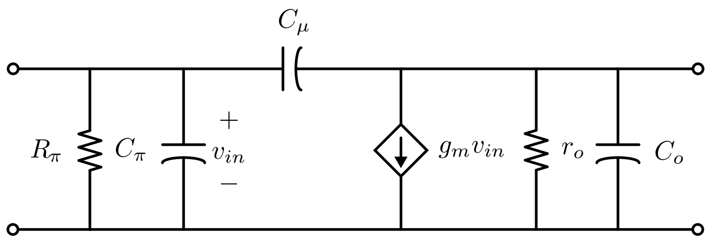


Figure 24: The simple hybrid-pi model for a transistor.

The hybrid-pi model for a transistor is shown in Fig. 24. Under what conditions is this two-port active? The hybrid matrix is given by

$$H(s) = \frac{1}{G_\pi + s(C_\pi + C_\mu)} \begin{pmatrix} 1 & sC_\mu \\ g_m - sC_\mu & q(s) \end{pmatrix} \tag{57}$$

$$q(s) = (G_\pi + sC_\pi)(G_0 + sC_\mu) + sC_\mu(G_\pi + g_m) \tag{58}$$

Applying the condition for passivity we arrive at

$$4G_\pi G_0 \geq g_m^2 \tag{59}$$

The above equation is either satisfied for the two-port or not, regardless of frequency. Thus our analysis shows that the hybrid-pi model is not physical. We know from experience that real two-ports are active up to some frequency f_{max}.

5.4 Mason's Invariant U Function

In 1954, Samuel Mason discovered the function U given by [6]

$$U = \frac{|k_{21} - k_{12}|^2}{4(\Re(k_{11})\Re(k_{22}) - \Re(k_{12})\Re(k_{21}))} \tag{60}$$

For the hybrid matrix formulation (H or G), the U function is given by

$$U = \frac{|k_{21} + k_{12}|^2}{4(\Re(k_{11})\Re(k_{22}) + \Re(k_{12})\Re(k_{21}))} \tag{61}$$

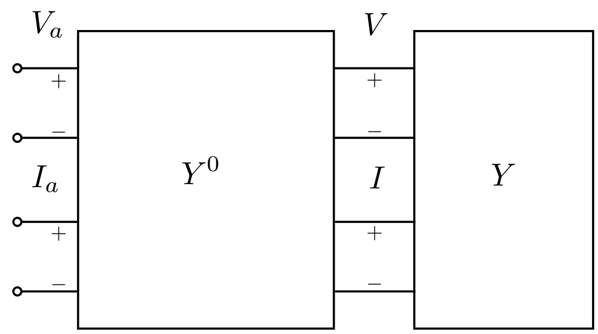


Figure 25: A general two-port described by Y is embedded into a lossless, reciprocal four-port device described by the matrix Y^0.

where k_{ij} are the two-port Y, Z, H, or G parameters.

This function is invariant under lossless reciprocal embeddings. Stated differently, any two-port can be *embedded* into a lossless and reciprocal circuit and the resulting two-port will have the same U function. This is a very important property, because this invariant property does not depend on any lossless matching circuitry that we employ before or after the two-port, or any lossless feedback.

5.4.1 Properties of U

The invariant property is shown in Fig. 25. The U of the original two-port is the same as U_a of the overall two-port when a four port lossless reciprocal four-port is added.

The U function has several important properties:

1. If $U > 1$, the two-port is active. Otherwise, if $U \leq 1$, the two-port is passive.
2. U is the maximum unilateral power gain of a device under a lossless reciprocal embedding.
3. U is the maximum gain of a three-terminal device regardless of the common terminal.

With regards to the previous diagram, any lossless reciprocal embedding can be seen as an interconnection of the original two-port to a four-port, with the following block admittance matrix [7]

$$\begin{pmatrix} I_a \\ -I \end{pmatrix} = \begin{pmatrix} Y_{11}^0 & Y_{12}^0 \\ Y_{21}^0 & Y_{22}^0 \end{pmatrix} \begin{pmatrix} V_a \\ V \end{pmatrix} \tag{62}$$

Note that Y_{ij} is a 2×2 imaginary symmetric sub-matrix

$$Y_{jk}^0 = jB_{jk} \tag{63}$$

$$B_{jk} = B_{kj}^T \tag{64}$$

Since $I = YV$, we can solve for V from the second equation

$$-I = Y_{21}^0 V_a + Y_{22}^0 V = -YV \tag{65}$$

$$V = -(Y + Y_{22}^0)^{-1} Y_{21}^0 V_a \tag{66}$$

From the first equation we have the composite two-port matrix

$$I_a = (Y_{11}^0 - Y_{12}^0 (Y + Y_{22}^0)^{-1} Y_{21}^0) V_a = Y_a V_a \tag{67}$$

By definition, the U function is given by

$$U = \frac{\det(Y_a - Y_a^T)}{\det(Y_a + Y_a^*)} \tag{68}$$

Note that Y_a can be written as

$$Y_a = jB_{11} - jB_{12}(Y + jB_{22})^{-1} jB_{12}^T \tag{69}$$

$$Y_a = jB_{11} + B_{12}(Y + jB_{22})^{-1} B_{12}^T \tag{70}$$

Focus on the denominator of U

$$Y_a + Y_a^* = B_{12}(W^{-1} + (W^*)^{-1}) B_{12}^T \tag{71}$$

where $W = Y + Y_{22}^0 = Y + jB_{22}$. Factoring W^{-1} from the left and $(W^*)^{-1}$ from the right, we have

$$= B_{12} W^{-1} (W^* + W)(W^*)^{-1} B_{12}^T \tag{72}$$

But $W + W^* = Y + Y^*$ resulting in

$$Y_a + Y_a^* = B_{12} W^{-1} (Y + Y^*)(W^*)^{-1} B_{12}^T \tag{73}$$

In a like manner, one can show that

$$Y_a - Y_a^T = B_{12} W^{-1} (Y^T - Y)(W^*)^{-1} B_{12}^T \tag{74}$$

Taking the determinants and ratios

$$\det(Y_a + Y_a^*) = \frac{(\det B_{12})^2 \det(Y + Y^*)}{(\det W)^2} \tag{75}$$

$$\det(Y_a - Y_a^T) = \frac{(\det B_{12})^2 \det(Y^T - Y)}{(\det W)^2} \tag{76}$$

$$U = \frac{\det(Y_a - Y_a^T)}{\det(Y_a + Y_a^*)} = \frac{\det(Y - Y^T)}{\det(Y + Y^*)} \tag{77}$$

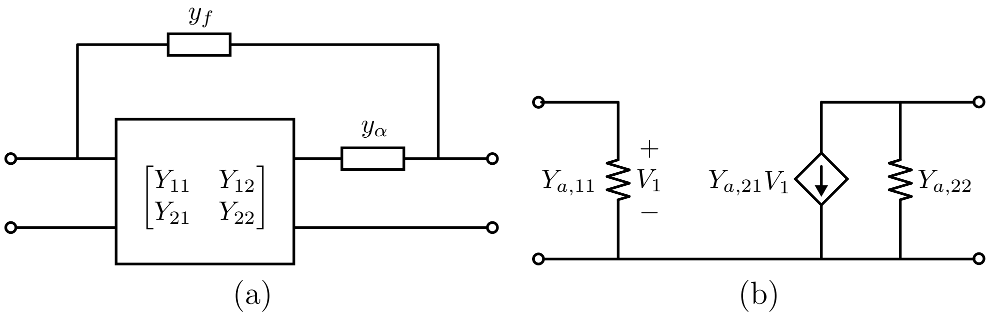


Figure 26: (a) A general two-port can be *unilaterized* by adding lossless feedback elements y_f and y_α. (b) The equivalent circuit for the unilaterized two-port.

5.4.2 Maximum Unilateral Gain

Consider Fig. 26a, a feedback structure where y_f and y_α are lossless reactances. We can derive the overall two-port equations by a cascade connection followed by a shunt connection of two-ports

$$Y_a = \frac{y_\alpha}{y_\alpha + y_{22}} \begin{bmatrix} y_{11} + \Delta_y / y_\alpha & y_{12} \\ y_{21} & y_{22} \end{bmatrix} + \begin{bmatrix} y_f & -y_f \\ -y_f & y_f \end{bmatrix} \tag{78}$$

To unilaterize the device, we select

$$y_f = \frac{y_{12} y_\alpha}{y_{22} + y_\alpha} \tag{79}$$

We can solve for b_α and b_f

$$b_f = \Im(y_{12}) - \frac{\Re(y_{12})}{\Re(y_{22})} \Im(y_{22}) \tag{80}$$

$$b_\alpha = b_f \frac{\Re(y_{22})}{\Re(y_{12})} \tag{81}$$

It can be shown that the overall Y_a matrix is given by

$$Y_a = \frac{j\Im(y_{22}^* y_{12})}{y_{12}\Re(y_{22})} \begin{bmatrix} y_{11} + y_{12} - j\frac{\Delta_y \Re(y_{12})}{\Im(y_{22}^* y_{12})} & 0 \\ y_{21} - y_{12} & y_{22} + y_{12} \end{bmatrix} \tag{82}$$

5.4.3 Unilaterized Two-Port

The two-port equivalent circuit under unilaterization is shown in Fig. 26b. Notice now that the maximum power gain of this circuit is given by

$$G_{U,max} = \frac{|Y_{a_{21}}|^2}{4\Re(Y_{a_{11}})\Re(Y_{a_{22}})} = U_a \tag{83}$$

We can now attribute physical significance to U_a as the maximum unilateral gain. Furthermore, due to the invariance of U, $U_a = U$ for the original two-port network. It's important to note that *any* unilaterization scheme will yield the same maximum power! Thus U is a good metric for the device.

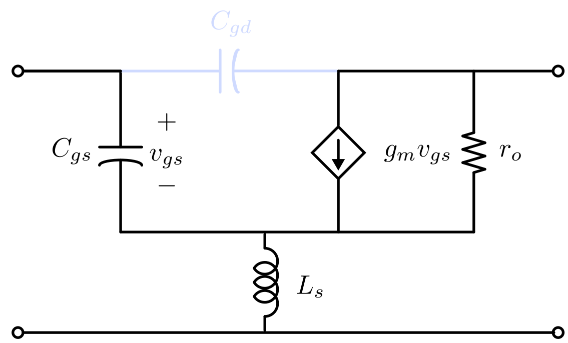


Figure 27: A FET with inductive degeneration.

Single Stage Feedback Revisited

With new tools at hand, let's revisit the problem of inductive and shunt feedback amplifiers.

5.4.4 Inductive Degeneration

Although $Z_{12} \approx 0$ for a FET at low frequency, the input impedance is purely capacitive. To introduce a real component, we found that inductive degeneration can be employed, shown schematically in Fig. 27. The Z matrix for the inductor is simply

$$Z_L = j\omega L_s \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{84}$$

Adding the Z matrix (due to series connection) to the Z matrix of the FET

$$Z = \begin{bmatrix} j\omega L_s + \frac{1}{j\omega C_{gs}} & j\omega L_s \\ j\omega L_s - \frac{g_m r_o}{j\omega C_{gs}} & r_o + j\omega L_s \end{bmatrix} \tag{85}$$

This feedback introduces a Z_{12} and thus the stability must be carefully examined

$$K = \frac{2 \cdot 0 \cdot r_o - \left(-\omega^2 L_s^2 - \frac{g_m L_s r_o}{C_{gs}}\right)}{\omega^2 L_s^2 + \frac{g_m r_o L_s}{C_{gs}}} = 1 \tag{86}$$

We see that this circuit is unconditionally stable. More importantly, the stability factor is frequency independent. In reality parasitics can destabilize the transistor.

The maximum gain is thus given by

$$G_{max} = \left|\frac{Z_{21}}{Z_{12}}\right| \left(K - \sqrt{K^2 - 1}\right) = \left|\frac{Z_{21}}{Z_{12}}\right| \tag{87}$$

$$= \frac{\omega L_s + \frac{g_m r_o}{\omega C_{gs}}}{\omega L_s} = 1 + \frac{g_m r_o}{\omega^2 L_s C_{gs}} \tag{88}$$

$$= 1 + \left(\frac{\omega_T}{\omega_0}\right)^2 \left(\frac{r_o}{\omega_T L_s}\right) \tag{89}$$

The synthesized real input resistance is given by $\omega_T L_s$, and so the last term is the ratio of r_o/R_S under matched conditions.

5.4.5 Capacitive Degeneration

A capacitively degenerated transistor is an important building block for Colpitts oscillators, where instability is desired. Using the same approach, the Z matrix for capacitive degeneration is given by

$$Z = \begin{bmatrix} \frac{1}{j\omega C_s} + \frac{1}{j\omega C_{gs}} & \frac{1}{j\omega C_s} \\ \frac{1}{j\omega C_s} - \frac{g_m r_o}{j\omega C_{gs}} & r_o + \frac{1}{j\omega C_s} \end{bmatrix} \tag{90}$$

The stability factor is given by

$$K = \frac{2 \cdot 0 \cdot r_o - \left(\frac{g_m r_o}{\omega^2 C_s C_{gs}} - \frac{1}{\omega^2 C_s^2} \right)}{\left| \frac{g_m r_o}{\omega^2 C_s C_{gs}} - \frac{1}{\omega^2 C_s^2} \right|} \tag{91}$$

Note this is simply

$$K = \frac{-a+b}{|a-b|} = \begin{cases} \frac{b-a}{a-b} < 0 & a > b \\ \frac{b-a}{b-a} = 1 & b < a \end{cases} \tag{92}$$

The condition for stability is therefore

$$\frac{g_m r_o}{C_{gs}} > \frac{1}{C_s} \tag{93}$$

So far we have dealt with $K > 0$. Suppose that $|\Delta| > 1$. We know that for $0 < K < 1$ the two-port is conditionally stable. In other words, the stability circle intersects with the unit circle with the overlap (usually) corresponding to the unstable region. Instability can also occur if $K > 1$ and $|\Delta| > 1$, but this is less common (occurs with feedback).

On the other hand, if $-1 < K < 0$, one can show graphically that the entire unit circle on the Smith Chart is unstable. In other words, the stability circle does not intersect with the unit circle or the instability circle contains the entire circle.

Unintentional capacitive degeneration is very common. For instance a common drain (source follower) driving a capacitive load may have stability problems. Likewise, a cascode amplifier may become unstable at high frequencies since the g_m input stage presents capacitive degeneration to the cascode device at high frequency.

5.4.6 Resistive Degeneration

Resistive degeneration is commonly employed to stabilize the bias point of a transistor. The overall Z matrix is given by

$$Z = \begin{bmatrix} R_s + \frac{1}{j\omega C_{gs}} & R_s \\ R_s - \frac{g_m r_o}{j\omega C_{gs}} & r_o + R_s \end{bmatrix} \tag{94}$$

The K factor is computed as before

$$K = \frac{2R_s(r_o + R_s) - R_s^2}{R_s \sqrt{R_s^2 + \frac{g_m^2 r_o^2}{\omega^2 C_{gs}^2}}} \tag{95}$$

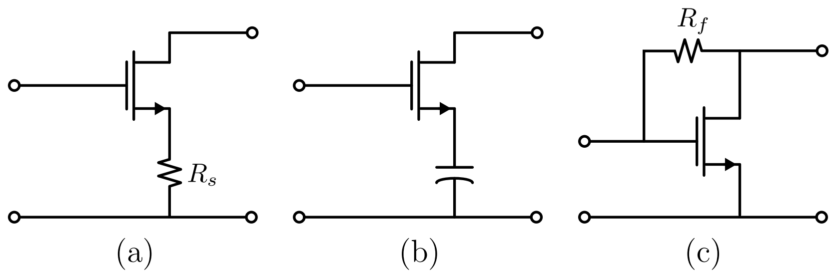


Figure 28: Common source amplifier with (a) capacitive degeneration, (b) resistive degeneration, (c) shunt feedback.

At low frequencies, we have

$$K = \frac{2r_o + R_s}{\frac{g_m r_o}{\omega C_{gs}}} \approx \frac{2\omega C_{gs}}{g_m} = \frac{2\omega}{\omega_T} < 1 \tag{96}$$

5.4.7 Shunt Feedback

We have seen that shunt feedback is a common broadband matching approach. Now working with the Y matrix of the transistor (simplified as before)

$$Y_{fet} = \begin{bmatrix} j\omega C_{gs} & 0 \\ g_m & G_o + j\omega C_{ds} \end{bmatrix} \tag{97}$$

The feedback element has a Y matrix

$$Y_f = G_f \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \tag{98}$$

And thus the overall amplifier Y matrix is given by

$$Y = \begin{bmatrix} G_f + j\omega C_{gs} & -G_f \\ g_m - G_f & G_f + G_o + j\omega C_{ds} \end{bmatrix} \tag{99}$$

The stability factor for the shunt feedback amplifier is given by

$$K = \frac{2G_f(G_o + G_f) - G_f(G_f - g_m)}{G_f|g_m - G_f|} \tag{100}$$

Suppose that $g_m R_f > 1$

$$= \frac{g_m + G_f}{g_m - G_f} = \frac{g_m R_f + 1}{g_m R_f - 1} > 1 \tag{101}$$

The choice of R_f and g_m is governed by the current consumption, power gain, and impedance matching. For a bi-conjugate match

$$G_{max} = \left|\frac{Y_{21}}{Y_{12}}\right| \left(K - \sqrt{K^2 - 1}\right) \tag{102}$$

$$= \frac{g_m - G_f}{G_f}\left(\left(\frac{g_m R_f + 1}{g_m R_f - 1}\right) - \sqrt{\left(\frac{g_m R_f + 1}{g_m R_f - 1}\right)^2 - 1}\right) = \left(1 - \sqrt{g_m R_F}\right)^2 \tag{103}$$

The input admittance is calculated as follows

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_L} \tag{104}$$

$$= j\omega C_{gs} + G_f - \frac{-G_f(g_m - G_f)}{G_o + G_f + G_L + j\omega C_{ds}} \tag{105}$$

$$= j\omega C_{gs} + G_f + \frac{G_f(g_m - G_f)(G_o + G_f + G_L - j\omega C_{ds})}{(G_o + G_f + G_L)^2 + \omega^2 C_{ds}^2} \tag{106}$$

At lower frequencies, $\omega < \frac{1}{C_{ds} R_f || R_L}$ we have (neglecting G_o)

$$\Re(Y_{in}) = G_f + \frac{G_f(g_m - G_f)}{G_f + G_L} \tag{107}$$

$$= \frac{1 + g_m R_L}{R_F + R_L} \tag{108}$$

$$\Im(Y_{in}) = \omega\left(C_{gs} - \frac{C_{ds}}{1 + \frac{R_f}{R_L}}\right) \tag{109}$$

6 Transistor Figures of Merit

A common figure of merit to characterize transistors is the device unity gain frequency, f_T, which are connected to the fundamental device physics. But RF device characterization is based upon f_{max}, or the maximum frequency where we can extract power gain from the device. Essentially, beyond the f_{max} frequency, the device is passive and it cannot be used to build an amplifier with power gain. Likewise, beyond f_{max} one cannot build an oscillator from an amplifier since oscillators need nearly infinite power gain, usually realized through feedback.[1] If a device does not have power gain, it certainly cannot have infinite power gain with feedback, and so the f_{max} frequency also corresponds to the maximum frequency of oscillation.

By definition, therefore, the frequency point where G_{max} crosses unity is the f_{max} of a two-port. Recall that G_{max} is only defined when the transistor is unconditionally stable, or $K > 1$. If $K < 1$, G_{max} is undefined and we usually speak of the maximum stable gain G_{MSG}, which corresponds to the maximum gain when the transistor is stabilized by adding positive conductance at the input and/or output ports so that $K = 1$.

In practice, the device f_{max} is usually estimated by plotting the device maximum unilateral power gain, or Mason's Gain U, and either observing or extrapolating the unity gain frequency point. This procedure should be performed with care and extrapolations should

[1] Nearly infinite because in any real circuit there is noise and thus the oscillator power gain is extremely large, but not infinite.

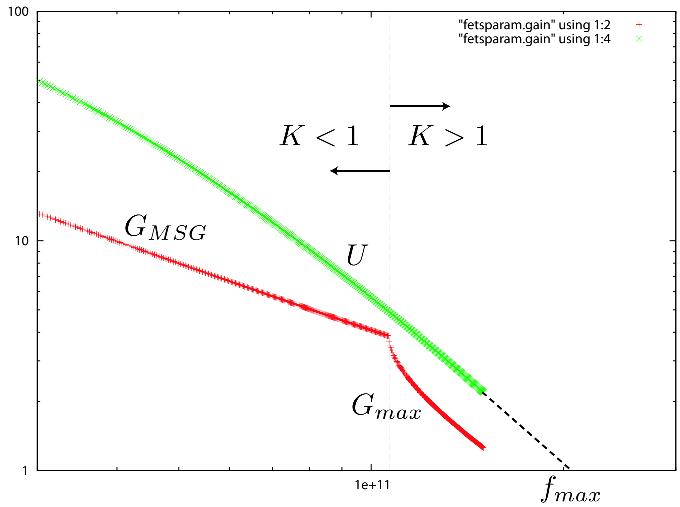


Figure 29: The various gain curves for a two-port device. The device is unstable at low frequency, $K < 1$, and thus we plot the G_{MSG} in this region. At the breakpoint, the device is stable. At high frequency the device is stable and we plot the G_{max} curve. The maximum unilateral gain U is also shown.

be avoided for maximal accuracy. If data is not available (e.g. above 100 GHz), it's better to model the device with an equivalent circuit up to the limits of measurements and then to use the f_{max} from directly evaluating the model up to the point when the power gain crosses unity.

In Fig. 29, the device G_{MSG} is plotted for low frequencies where $K < 1$. At the breakpoint, $K = 1$ and the device is unconditionally stable and thus G_{max} is plotted. Note that the U curve is always larger than G_{max} but both curves cross 0 dB together. At this point, the f_{max} of the device, the two-port becomes passive. f_{max} is a good metric for characterizing a three terminal device with a common-terminal, such as a transistor. Since U is invariant to the common terminal, a common-gate amplifier has the same U as a common-source amplifier.

Using the unilateral gain U, the f_{max} of a BJT transistor can be estimated by

$$f_{max} \approx \sqrt{\frac{f_T}{8\pi r_b C_\mu}} \tag{110}$$

where the base resistance r_b and feedback capacitance C_μ are seen to set the ultimate frequency of operation for a device. It's interesting to observe that in most low frequency design, both of these effects are ignored with negligible error. But design close to the limits of a device f_{max} requires careful modeling of all parasitic feedback and loss mechanisms. In particular, the distributed nature of the feedback C_μ into r_b requires a sectional model.

The cross section of a MOSFET device is shown in Fig. 30. The f_{max} of a modern FET transistor can be estimated by [8]

$$f_{max} \approx \frac{f_T}{2\sqrt{R_g\left(g_m C_{gd}/C_{gg}\right) + \left(R_g + r_{ch} + R_S\right) g_{ds}}} \tag{111}$$

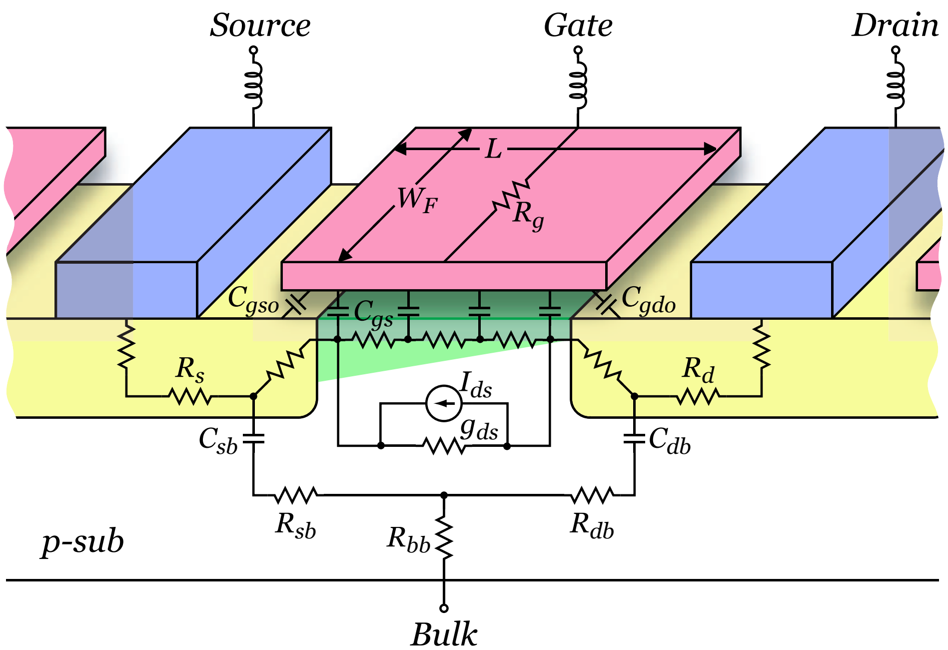


Figure 30: The cross section of a FET device showing the important high frequency parasitics. *Courtesy of Chinh Doan.*

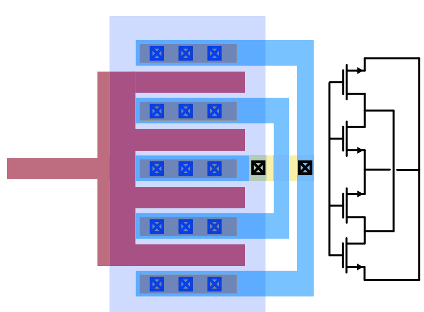


Figure 31: A high frequency multi-finger FET layout minimizes the poly gate resistance.

In contrast to the device f_T, the f_{max} is a strong function of the losses in the device. As MOS technology scaling continues, f_T improves almost proportional to channel length due to velocity saturation. But shorter channel devices may have higher gate, source and drain losses. It is interesting to note that the effect of gate resistance on f_{max} can be reduced by scaling the width of the transistor W through a multi-finger layout, as shown in Fig. 31. The drain and source resistances, though, do not scale and pose a challenge for next-generation technologies. This is in stark contrast to MESFET devices where a low resistance metal gate is employed. In deeply scaled MOS technology, metal gate work-function engineering may replace doping as a means to set the threshold voltage of a device, leading to enhanced RF performance.

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