Smith Charts

Appendix A

Scattering Parameters & Smith Charts

Scattering Parameters

Many of the readers of this book are analog engineers who are not well versed in s-parameters and Smith charts. While it is possible to get a lot of value from the book without understanding these things, an understanding is necessary if the output from Genesys is to be understood. For that reason, I am going to provide a brief overview of scattering parameters in this section. If you are already familiar with s-parameters, you may skip to the next section.

The fact that the average analog engineer is unfamiliar with these concepts (and has probably never used a program like Genesys before), is not a good reason not to learn and use these techniques. I am quite certain that these tools are very useful even if you are designing low frequency circuits. A few of the many reasons I make this statement are: 1) The transistors you use don't know that they are supposed to work only at audio frequencies. They are perfectly happy to oscillate at many GHz if allowed. 2) Genesys (and other programs like it) contain modules that enable you to do EM modeling of things like circuit boards. This can be quite useful in making your circuit EMI hardened (a bane of many analog circuits). 3) These tools provide new insights into analog design broadening your knowledge and capabilities; insight that may help keep you ahead of your competitors. So while understanding s-parameters and having access to tools like Genesys is critical for the RF and high frequency, wide bandwidth analog designer, they are extremely useful for the low frequency analog designer as well.

Scattering parameters are all about power; both reflected and incident in a linear two port system. It assumes that the system must be treated like a transmission line system; lumped elements no longer adequately describe the system.

For the following analysis, refer to Figure: A-1.

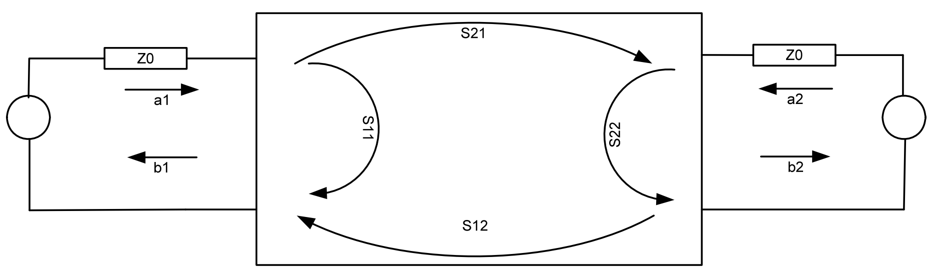

Figure: A-1 S Parameter Two Port Model

The s-parameter definition is:

$$b_1 = s_{11}a_1 + s_{12}a_2$$

$$b_2 = s_{21}a_1 + s_{22}a_2$$

Lets look at the independent and dependent variables, a and b.

The independent variables, a_1 and a_2 are normalized incident voltages defined as:

$$a_1 = \frac{V_1 + I_1 Z_0}{2\sqrt{Z_0}} = \frac{\text{voltage wave incident on port 1}}{\sqrt{Z_0}} = \frac{V_{i1}}{\sqrt{Z_0}}$$

$$a_2 = \frac{V_2 + I_2 Z_0}{2\sqrt{Z_0}} = \frac{\text{voltage wave incident on port 2}}{\sqrt{Z_0}} = \frac{V_{i2}}{\sqrt{Z_0}}$$

The dependent variables, b_1 and b_2 are normalized reflected variables defined as:

$$b_1 = \frac{V_1 - I_1 Z_0}{2\sqrt{Z_0}} = \frac{\text{voltage wave reflected from port 1}}{\sqrt{Z_0}} = \frac{V_{r1}}{\sqrt{Z_0}}$$

$$b_2 = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}} = \frac{\text{voltage wave reflected from port 2}}{\sqrt{Z_0}} = \frac{V_{r2}}{\sqrt{Z_0}}$$

Restating the above equations for a and b gives:

$|a_1|^2$ = Power incident on the input of the network.
= Power available from a source impedance Z_0.

$|a_2|^2$ = Power incident on the output of the network
= Power reflected from the load.

$|b_1|^2$ = Power reflected from the input port of the network.
= Power available from a Z_0 source minus the power delievered to the input of the network.

$|b_2|^2$ = Power reflected from the output port of the network.
= Power incident on the load.
= Power that would be delivered to a Z_0 load.

While a and b represent transmission and reflections of power or voltage at the input and output of the 2-port, what is the meaning of the s-parameter coefficients? These definitions are:

$$s_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \left.\frac{V_{\text{reflected at port 1}}}{V_{\text{towards port 1}}}\right|_{a_2=0}$$

s_{11} = input reflection coefficient with the output port terminated by a matched load (Z_L=Z_0 sets $a_2 = 0$)

$$s_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} = \left.\frac{V_{\text{reflected at port 2}}}{V_{\text{towards port 2}}}\right|_{a_1=0}$$

s_{22} = output reflection coefficient with the input port terminated by a matched load (Z_s=Z_0 sets a_1 = 0)

$$s_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_{\text{out of port 2}}}{V_{\text{towards port 1}}}\right|_{a_2=0}$$

s_{21} = forward transmission (insertion) gain with the output port terminated by a matched load (Z_L=Z_0 sets a_2 = 0)

$$s_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} = \left.\frac{V_{\text{out of port 1}}}{V_{\text{towards port 2}}}\right|_{a_1=0}$$

s_{12} = reverse transmission (insertion) gain with the input port terminated by a matched load (Z_s=Z_0 sets a_1 = 0)

Or

Squaring the s-parameters gives us the following:

$$\left|s_{11}\right|^2 = \frac{\text{Power reflected from the network input}}{\text{Power incident on the network input}}$$

$$\left|s_{22}\right|^2 = \frac{\text{Power reflected from the network output}}{\text{Power incident on the network output}}$$

$$\left|s_{21}\right|^2 = \frac{\text{Power delivered to a } Z_0 \text{ load}}{\text{Power available from } Z_0 \text{ source}}$$
$$= \text{Transducer power gain with both load and source having an impedance of } Z_0$$

$$\left|s_{12}\right|^2 = \text{Reverse transducer power gain with } Z_0 \text{ load and source.}$$

Z_s and Z_L are the source and load terminating impedances respectively. Z_0 is called the reference impedance for the two port. While it doesn't have to be, in this book we will always define Z_0 to be positive and real. In most cases, we will make the reference impedance 50 ohms.

Given the discussion on reflection coefficients and matched loads, s-parameters clearly have something to do with transmission lines. In particular, at very high frequencies, "lumped elements" no longer look "lumped" but instead begin to look like some kind of distributed transmission line. In addition, other two port parameters (z, y, ABCD, etc) all rely on the ability to create either an ideal short or an ideal open (or both) at one of the ports in order to measure the parameter. At high frequencies; this is not possible because of inductance preventing a complete short, and capacitance preventing a complete open. I should point out, that it is possible to make the measurement using s-parameters, and then mathematically convert to one of the other 2-port definitions. Such conversion equations do exist.

Now let's look at deeper meanings for these parameters.

For those of you who don't remember, the definition of reflection coefficient when applied to a transmission line is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Γ_L ranges from -1 to +1 (when the load is either infinity or zero respectively). When $Z_0 = Z_L$, $\Gamma_L = 0$ and there are no transmission line reflections allowing maximum power to be delivered to the load.
Given this definition for Γ and the previous definitions for s_{11} and s_{22} we arrive at the following definitions:

$$S_{11} = \frac{b_1}{a_1} = \frac{\frac{V_1}{I_1} - Z_0}{\frac{V_1}{I_1} + Z_0} = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \Gamma_{in} \text{ (the reflection coefficient at the input port)}$$

Where V_1 is the voltage at port 1, I_1 is the current at port 1 and Z_1 is the input impedance at port 1.

Using this definition for s_{11}, we can write an expression for VSWR at the input port:

$$VSWR\big|_{\text{input port}} = \frac{1 + |s_{11}|}{1 - |s_{11}|}$$

Similarly we can write the definition of s_{22} as:

$$S_{22} = \frac{b_2}{a_2} = \frac{\frac{V_2}{I_2} - Z_0}{\frac{V_2}{I_2} + Z_0} = \frac{Z_2 - Z_0}{Z_2 + Z_0} = \Gamma_{out} \text{ (the reflection coefficient at the output)}$$

It is clear that both s_{11} and s_{22} are transmission line reflection coefficients. Because Smith charts were developed for studying transmission lines; they are perfect for displaying these parameters.

$$VSWR\big|_{\text{output port}} = \frac{1 + |s_{22}|}{1 - |s_{22}|}$$

We can take this one more step and rearrange these equations to give the input and output impedance in terms of s_{11}, s_{22}, and Z_0. Doing this gives:

$$z_1 = z_0 \frac{1 + s_{11}}{1 - s_{11}}$$

$$z_2 = z_0 \frac{1 + s_{22}}{1 - s_{22}}$$

Finally, we can define a parameter called insertion loss to be:

$$R_L = -20 \log |s_{11}|$$

What about s_{21} and s_{12}? Is there additional insight from these parameters? The answer is yes. Looking at s_{21} first, assume that we set a_2 =0. This sets

$I_2^+ = V_2^+ = 0$ where I^+ and V^+ represent the forward traveling wave components

By replacing the V_1 in the equation for s_{21} with the generator voltage less the voltage drop over the source impedance, Z_0 gives (for the case of matched loads only):

$s_{21} = \frac{2V_2}{V_{g1}}$ and thus specifies the forward voltage gain.

By a similar process, we find that s_{12} is equal to twice the reverse voltage gain.

The forward power gain is G_0 and is equal to:

$$G_0 = |s_{21}|^2 = \left| \frac{V_2}{V_{g1}/2} \right|^2$$

Looking at a Genesys example for the very simple passive network circuit shown Figure 0-1 will help clarify these ideas.

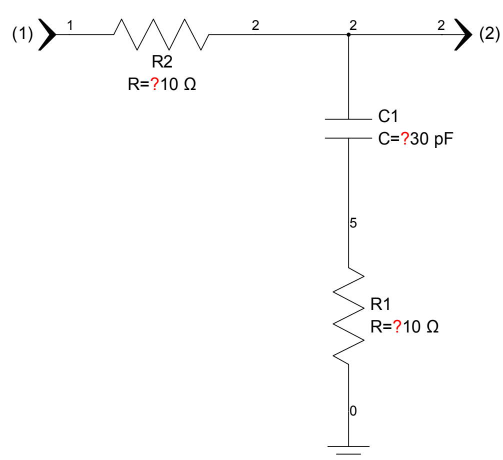

Figure 0-1 A Passive Circuit To Aid In Explaining S-Parameters

Port 1 and port 2 both have port impedances of 50 ohms. Call these impedances Z_s and Z_L. By inspection, at low frequencies C1 is open so the input impedance should be r_2 plus Z_L impedance = 10 +50 = 60 ohms. At high frequencies, C_1 is a short so the input impedance is R_2 plus the parallel combination of R_1 and Z_L which equals 10 ohms plus 8.333 ohms = 18.333 ohms.

Because C1 is open, the voltage gain from port 1 to port 2 at low frequencies is:

$$E_{v21} = \frac{Z_L}{Z_L + R_2} = \frac{50}{60} = 0.8333$$

At high frequencies, C_2 is a short and the voltage gain from port 1 to port 2 becomes:

$$E_{v21} = \frac{R_p}{R_2 + R_p} \text{ where } R_p = \frac{Z_L R_1}{Z_L + R_1}$$

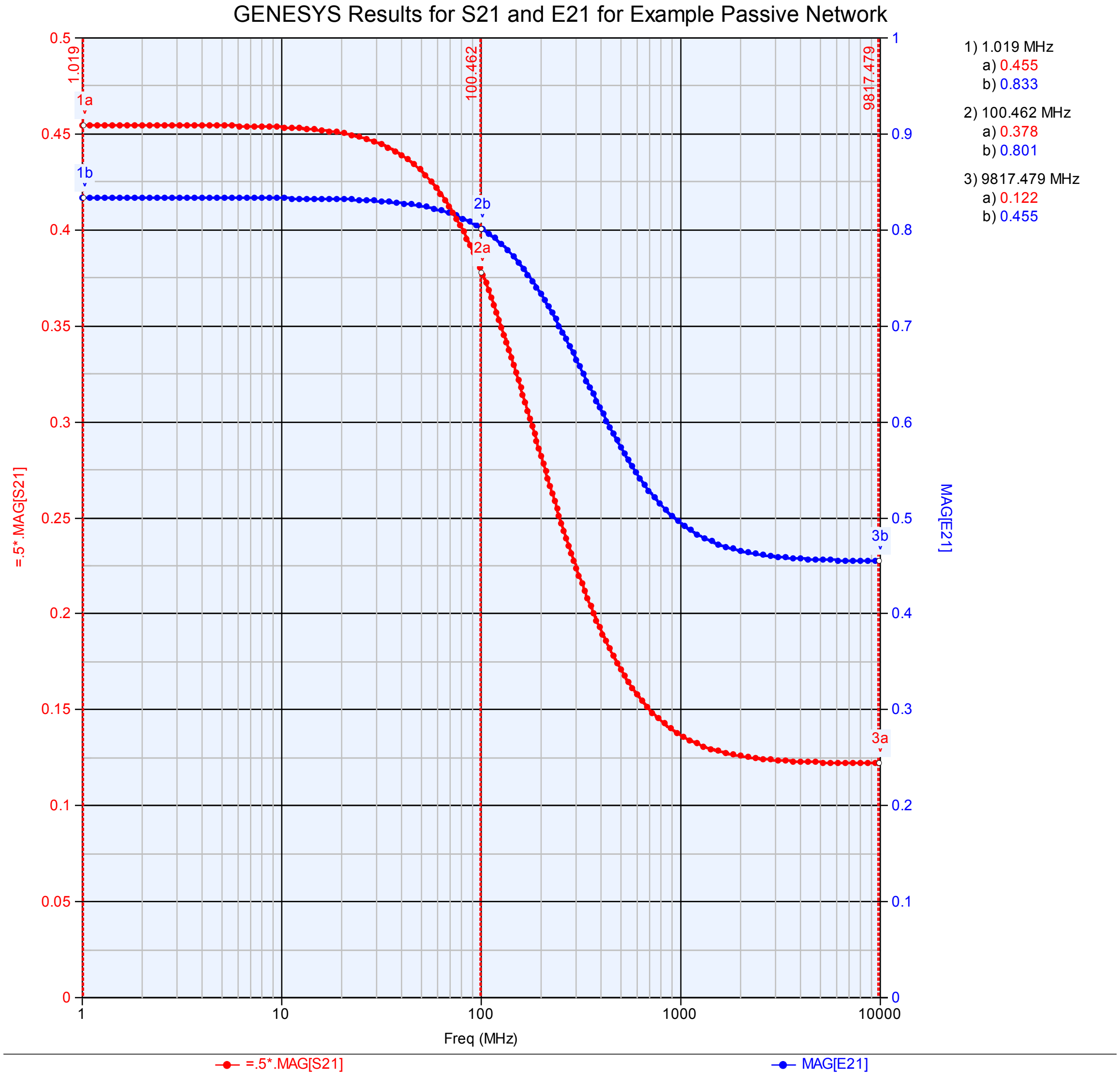

Figure 0-2: .5 *s_{22} And Voltage Gain Output From Genesys Passive Network Circuit

This gives a result of $E_{v21} = \dfrac{8.33}{8.33+10} = .454$ This definition for voltage gain is the voltage gain (E21) provided by Genesys in Figure 0-2.

We have stated that s_{21} is also a voltage gain. But it is referenced differently from the E_{v21}. In particular, it is the gain to the output port from the source voltage source. In other words, the input voltage is on the left side of the source impedance instead of the right side. This gives a new value for voltage gain that is equal to:

$$A_{v21} = \frac{Z_L}{Z_L + R_2 + Z_s} = \frac{50}{110} = .455 \text{ at low frequencies and}$$

$$A_{v21} = \frac{R_p}{R_2 + R_p + Z_s} \text{ where } R_p = \frac{Z_L R_1}{Z_L + R_1} \text{ at high frequencies}$$

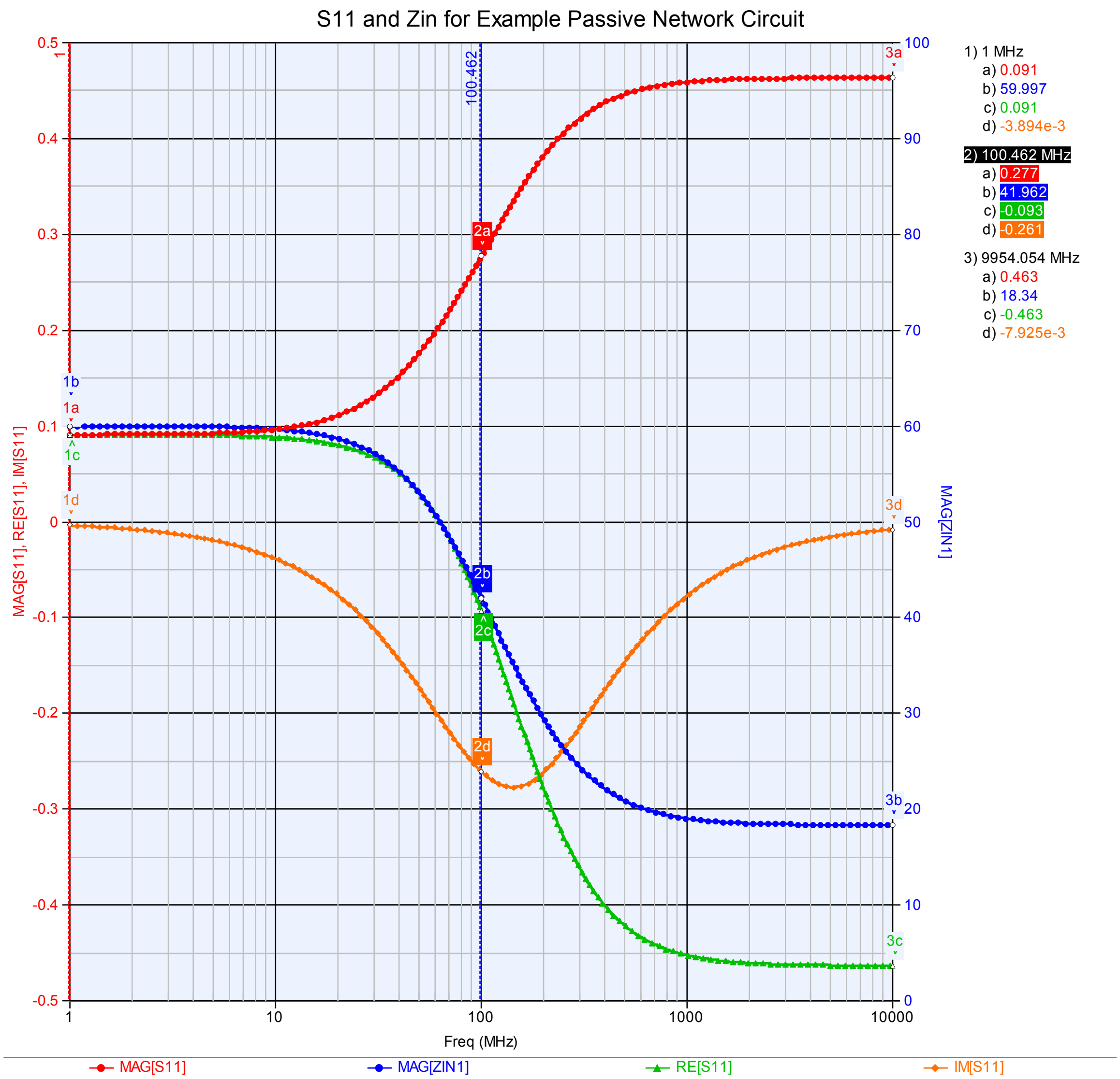

Figure 0-3: s_{11} and Zin for Example Passive Network

Giving $A_{v21} = \dfrac{8.33}{8.33+10+50} = .122$

Remember, s_{22} is twice this voltage gain. Figure 0-2 is a plot of s_{22} multiplied by .5 (and also a plot of E_{21}). This results in values that are identical at the low and high frequency end for the calculated vale of A_{v21} and $.5*s_{22}$. Thus proving our assertion s_{21} is related to voltage gain.

The magnitude of s_{11} and Zin is shown in Figure 0-3. Remember s_{11} is the reflection coefficient at the input port. As such it is related to Zin by the equation $z_1 = z_0 \dfrac{1+s_{11}}{1-s_{11}}$. Note that s_{11} is a complex number, meaning Zin is normally complex. In this example, at low frequencies, and high frequencies, the imaginary part of s_{11} is close to zero. This allows us to substitute into the z_1 equation only the real part of s_{11} and avoid complex arithmetic.

At low frequencies, we see that reap part of s_{11} is equal to .091 while at high frequencies it is -.463. Substituting these numbers into the equation for Z_1 we obtain:

$$z_{1low} = 50\frac{1+.091}{1-.091} = 50\frac{1.091}{.909} = 50*1.2 = 60$$

And

$$Z_{1high} = 50\frac{1-.463}{1+.463} = 50\frac{.537}{1.463} = 50*.367 = 18.35$$

Observe that these numbers are the same as the Genesys numbers for Zin

Smith Charts

Smith charts were invented to handle transmission line calculations before computers made life easier. Because of their unique nature, they are still one of the better ways to display data related to transmission lines. In particular, they map a complex rectangular impedance plane into a polar system that represents complex reflection coefficients. As such, they are perfect for representing s-parameters. Figure 0-4 shows a Smith chart displaying s_{11} for our example circuit of Figure 0-1.

Notice that the horizontal axis represents pure resistance with 0 at the far left and infinity at the far right. The center of the circle is 1 and represents the normalized reference impedance (normally 50 ohms). If a point lands on 1, it represents a 50 ohm resistor. Notice that at 1 MHz, a low frequency, s_{11} is on the real axis with a value of 1.2 frequencies. Multiplying 1.2 by 50 gives 60 ohms; the input impedance of the network at this frequency. This is easier than going through the calculation from the equation $z_1 = z_0\frac{1+s_{11}}{1-s_{11}}$; the Smith chart does the calculation for you. Similarly, at 10 GHz s_{11} is also on the real axis at about .365. Multiplying .365 by 50 gives 18.25; the input impedance to the network at 10 GHz. These are, within the error of reading the graph, the same as we obtained before.

Circles on the Smith chart represent constant resistance curves, while the arcs radiating out from the right side to the edge of the Smith chart represent reactance curves. Notice that at 108 MHz, s_{11} intercepts a constant resistance curve equal to .7 and a constant reactance curve of about -.41. Multiplying these numbers by 50 gives an impedance for our circuit at 108 MHz of 35 –j20.5 ohms. Had we taken the actual value for s_{11} of .112 –j.267 and plugged it into $z_1 = z_0\frac{1+s_{11}}{1-s_{11}}$, we would have obtained the same result.

Observe that points below the real axis representative capacitive circuits while points above the real axis will represent inductive reactance circuits. Any point on the edge of the Smith chart represents a pure reactance.

Finally, observe that increasing the frequency always moves the graph in a clockwise direction.

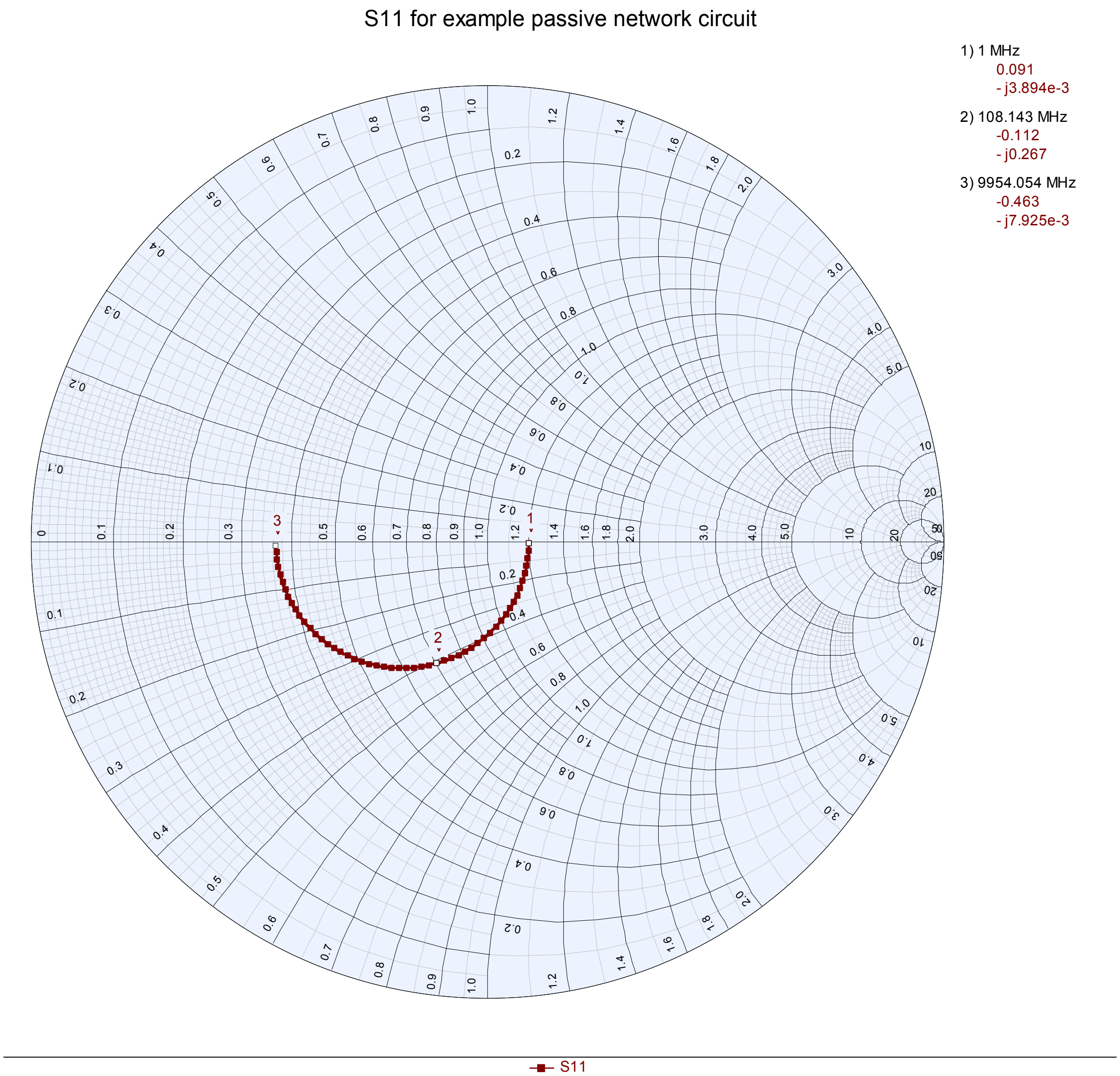

Figure 0-4: A Smith Chart Showing s_{11} for Our Example Circuit