

# 1 Introducing loop gain

## WHAT IS LOOP GAIN?

When analyzing an analog circuit with feedback, loop gain, denoted  $T(s)$ , is one of the most important parameters to find. This is because many other things in which we are interested are themselves functions of  $T(s)$ .

## FINDING THE LOOP GAIN

Finding  $T(s)$  can be hard because, in many cases, you have local feedback loops (e.g. due to  $C_{gd}$  of a common-source amplifier), residing within a global feedback loop (e.g. the feedback around an OTA). However, one method that generally works very well is *Return Ratio Analysis*. Though it looks like it involves several steps when written out, it is an entirely intuitive thing to do.

1. Remove a “primary” *dependent* source (e.g. a  $g_m$  element) and inject a test signal in its place.
2. Calculate the returned signal, which is what comes back out of the dependent source while you are injecting the test signal.
- 3.

$$T(s) = -\frac{\text{signal}_r}{\text{signal}_t}$$

Note that  $T(s)$  will usually be frequency dependent, and will often be large at low frequencies and small at high frequencies. Fig. 1 shows an example of this strategy. In this case, the primary dependent source is a  $g_m$  element. Therefore, it is removed and a test signal is injected in its place. All we have to do is calculate what the  $g_m$  element does in response to the test signal.

## RELATING LOOP GAIN TO CLOSED-LOOP PERFORMANCE

Once we compute  $T(s)$ , we are still not done determining the closed-loop performance of the circuit. To move us along in that direction, we start with Fig. 2.

To begin, solve for the transfer function,  $v_{out}/v_{in}$ .

$$\frac{v_{out}}{v_{in}} = \frac{a(s)}{1 + a(s)f(s)}$$

There are two modifications I wish to make to this equation. First, note that the term  $a(s)f(s)$  is the loop gain of this circuit (you could determine this using loop gain analysis), so we can replace this quantity by  $T(s)$ .

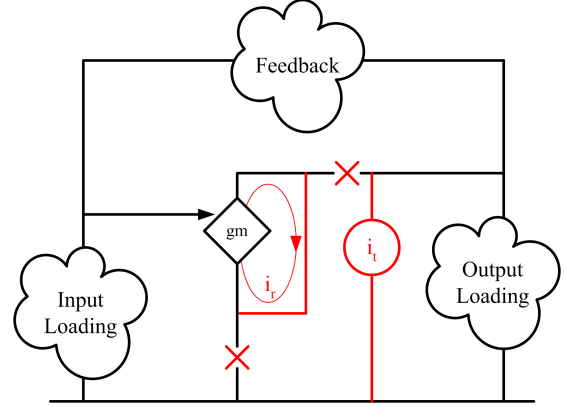


Figure 1: In this example, we find the loop gain by removing the  $g_m$  element, injecting a test signal, and measuring what comes back out of the  $g_m$  element

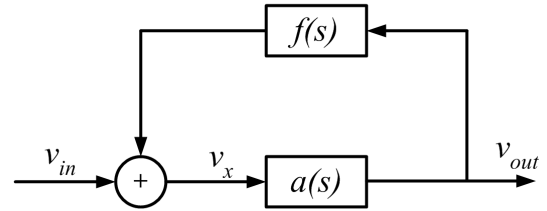


Figure 2: Generic feedback system

Secondly, I want to make this analysis more useful by recognizing that we usually want  $a(s)$  to be as large as possible (e.g. often thousands or millions). Assuming infinity is the ideal value of  $a(s)$ , it must follow that  $1/f(s)$  is the ideal closed-loop gain (e.g. 2 or something like that). Because of that, we will define  $A_\infty \triangleq 1/f(s)$ . Plugging back into the transfer function

$$\frac{v_{out}}{v_{in}} = A_\infty \frac{T(s)}{1 + T(s)} \quad (1)$$

So, when the loop gain,  $T(s)$ , is large, then the circuit will achieve the ideal gain of  $A_\infty$ .

The next aspect we want to account for (which is totally absent from Eqn. 1) is feedthrough. We define the passive feedthrough,  $d(s)$ , as the transfer function when all the active elements are removed. You can think of it as a path *around* the amplifier, as shown in Fig. 3.

Again we can compute the closed loop transfer func-

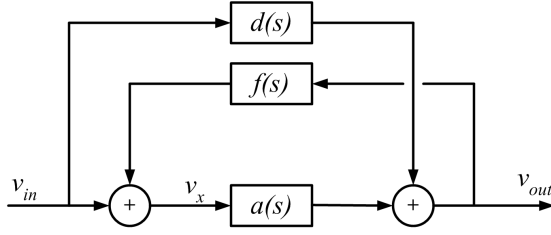


Figure 3: Generic feedback system with feedthrough

tion

$$\frac{v_{out}}{v_{in}} = H(s) = \frac{a(s)}{1 + a(s)f(s)} + \frac{d(s)}{1 + a(s)f(s)} \quad (2)$$

$$= A_{\infty} \frac{T(s)}{1 + T(s)} + \frac{d(s)}{1 + T(s)} \quad (3)$$

The block diagram in Fig. 3 is extremely simple, and determining what goes in each of these blocks for any real analog circuit would be very difficult. However, the closed loop transfer function in Eqn. 3 is actually pretty general. As seems intuitive, large  $T(s)$  helps achieve ideal gain and reduce feedthrough.

We also get a few other things for free. The low-frequency gain is

$$A_o = A_{\infty} \frac{T(0)}{1 + T(0)} + \frac{d(0)}{1 + T(0)}$$

The steady state error, which I will not derive, is approximately

$$\varepsilon \approx 1 - \frac{A_o}{A_{\infty}}$$

Finally, we define the return factor,  $\beta$ .

$$\beta = \frac{V_x}{V_{out}} = \frac{Z_i \| Z_s}{(Z_i \| Z_s) + Z_f}$$

Note that you don't actually need to know  $\beta$  in order to find the loop gain. But the  $v_x/v_{out}$  ratio comes up often enough that it helps to have a short way of writing it.

#### STABILITY (GENERAL CASE)

Although Eqn. 3 gives a complete behavioral description, it is actually more than we need to determine the stability of the circuit. Stability depends only on the autonomous behavior of  $T(s)$  (another good reason for finding it!), which we can analyze using the Bode criterion. The Bode criterion is not as

general a tool as Nyquist plots, but it is very easy to use. It says that if  $T(s)$  is itself stable (usually true for simple amplifier circuits), then the closed-loop system will be stable as long as  $|T(s)| \leq 1$  when  $\text{Phase}(T(s)) = -180^\circ$ .

Furthermore, we not only care *if* a system is stable, but *how stable* it is. Two common metrics that convey the degree of stability are phase margin and gain margin.

$$\text{PM} = 180^\circ - \text{Phase}[T(s)]|_{\omega=\omega_c}$$

Typically we want  $\text{PM} \approx 60 - 70^\circ$ .  $\omega_c$  will be defined later.

$$\text{GM} = \frac{1}{|T(j\omega)|}|_{\omega=\omega_{180}}$$

Typically we want  $\text{GM} \approx 3...5$

Note that sometimes, only one of these two quantities might be defined (e.g. if the gain never goes above 1 or if the phase never reaches 180).

#### BLACKMAN'S IMPEDANCE FORMULA

$T(s)$  can also be used to determine port impedances. However, there is a slight twist in that  $T(s)$  must be recomputed under two different port conditions:

$$Z_{\text{port}} = \left[ Z_{\text{port}} \right]_{k=0} \times \frac{1 + T(\text{port shorted})}{1 + T(\text{port open})} \quad (4)$$

In Eqn. 4, the term  $Z_{\text{port}}(k = 0)$  refers to the impedance seen at the port in question when the feedback loop is broken. This is simply the open-loop impedance.

## 2 Approximations when there is a dominant pole

Return ratio analysis and the Bode criterion are fairly general methods. However, many circuits in which we are interested have a dominant pole, and in that case we can make further simplifications. These are not necessary for *analysis* of circuits. For analysis,  $T(s)$  tells us almost everything we need to know. Dominant pole approximations are very useful for *design*. When designing a circuit, it is hard to

work with fifth-order equations, and dominant-pole approximations allow us treat the system as though it were almost first order. Let's see how this is done.

First, any low-pass, single-pole system can be expressed as

$$a(s) = a_0 \frac{1}{1 + \frac{s}{\omega_p}}$$

Where  $a_0$  is the low-frequency gain and  $\omega_p$  is the dominant pole frequency. When a higher order system has a dominant pole, this single-pole expression serves as an approximation for the true behavior. Using this expression, we can define the following terms:

Gain-bandwidth product

$$\text{GBW} = \text{DC Gain} \times \text{Bandwidth} = a_0 \omega_p$$

Unity Gain Frequency

$$\begin{aligned} \omega_c \quad \text{s.t.} \quad 1 &= \left| a_0 \frac{1}{1 + \frac{j\omega_c}{\omega_p}} \right| \\ &\approx \left| \frac{a_0}{\frac{j\omega_c}{\omega_p}} \right| \\ &= \left| \frac{a_0 \omega_p}{j\omega_c} \right| \\ \therefore \omega_c &\approx a_0 \omega_p = \text{GBW} \end{aligned}$$

Note that even though  $\omega_c$  has units of frequency, it is not the same thing as a pole. The dominant pole is at  $\omega_p$ , and can be determined using a method such as zero-value time constants. The crossover frequency is simply a point on the frequency axis where the dominant pole as rolled off enough to finally overcome all the gain that was available at low frequencies. All of this is shown in Fig. 4. Note that in this figure, the  $-3\text{dB}$  point of the closed loop system occurs at approximately  $\omega_c$ . This approximation is usually valid, allowing us to write:

$$\omega_{-3\text{dB}} \approx \omega_c$$

STABILITY (DOMINANT POLE CASE)

As with the general case, we are only concerned with the autonomous behavior of  $T(s)$ . If  $T(s)$  had a single pole, the phase margin (and the phase shift) would (both) be  $90^\circ$ . This can be seen in Fig. 4. With only a single pole, the phase of  $T(s)$  bottoms out at  $-90^\circ$ .

What happens if there is a non-dominant pole in addition to the dominant one? Fig. 5 shows a two-pole

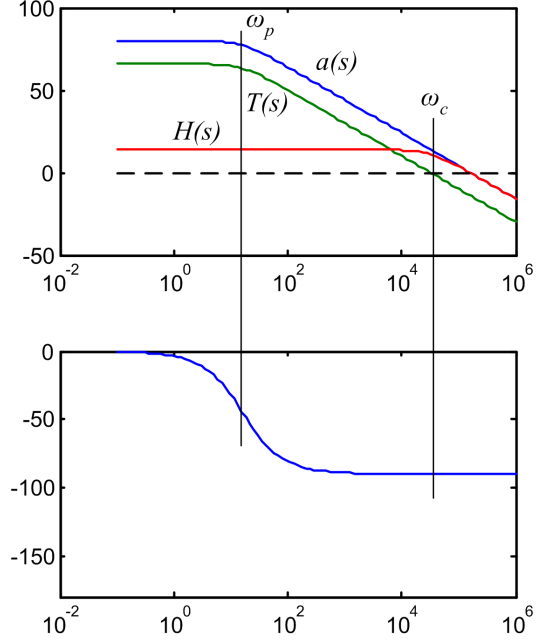


Figure 4: Circuit characteristics with a single pole

system with  $\omega_{p2} \approx \omega_c$ . There are two effects. First, the magnitude of  $T(s)$  rolls off a little faster, meaning that  $\omega_c$  is reduced slightly. But the bigger problem is that the phase margin has dropped from  $90^\circ$  to about  $45^\circ$ . This causes peaking in the closed-loop response,  $H(s)$ .

In higher order systems, non-dominant poles always cause additional phase shift, which reduces phase margin. The phase contribution from a non-dominant pole at  $\omega_c$  is  $-\tan^{-1}(\omega_c/\omega_{p2})$ . Therefore, PM becomes

$$\text{PM} = 90^\circ - \tan^{-1} \left( \frac{\omega_c}{\omega_{p2}} \right) = \tan^{-1} \left( \frac{\omega_{p2}}{\omega_c} \right)$$

We can maintain phase margin (PM) by keeping the onset of non-dominant poles well beyond  $\omega_c$ . The following table gives phase margin as a function of the ratio of  $\omega_{p2}$  to  $\omega_c$  (table is valid when there is only one non-dominant pole).

$\omega_{p2}/\omega_c$	PM
1	$45^\circ$
2	$63^\circ$
3	$72^\circ$
4	$76^\circ$
5	$79^\circ$

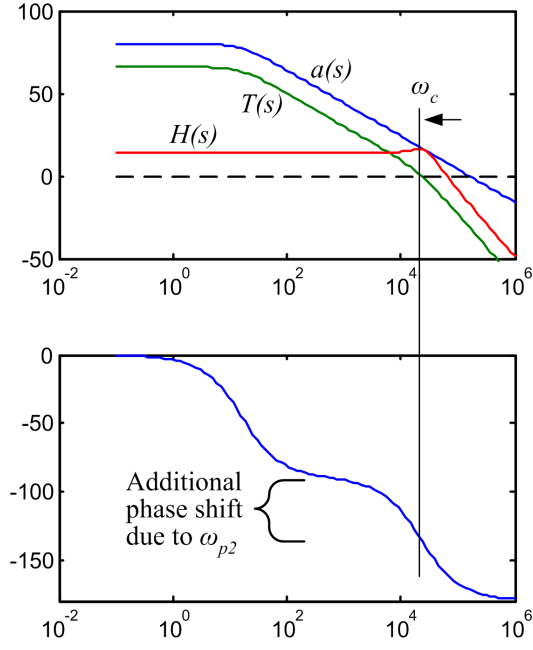


Figure 5: Circuit characteristics when  $\omega_{p2} \approx \omega_c$

As a demonstration, Fig. 6 shows that when we set  $\omega_{p2} \approx 3\omega_c$ , we have about  $72^\circ$  phase margin.

### 3 Summary of Equations

$$T(s) = -\left(\frac{v_r}{v_t}\right)(s), \quad (\text{voltage source})$$

$$= -\left(\frac{i_r}{i_t}\right)(s), \quad (\text{current source})$$

$$A(s) = A_\infty \frac{T(s)}{1 + T(s)} + d \frac{1}{1 + T(s)}$$

$$\varepsilon = 1 - \frac{A_o}{A_\infty}$$

$$A_o = A_\infty \frac{T(0)}{1 + T(0)}$$

$$Z_{\text{port}} = \left[ Z_{\text{port}} \Big|_{k=0} \right] \times \frac{1 + T(\text{port shorted})}{1 + T(\text{port open})}$$

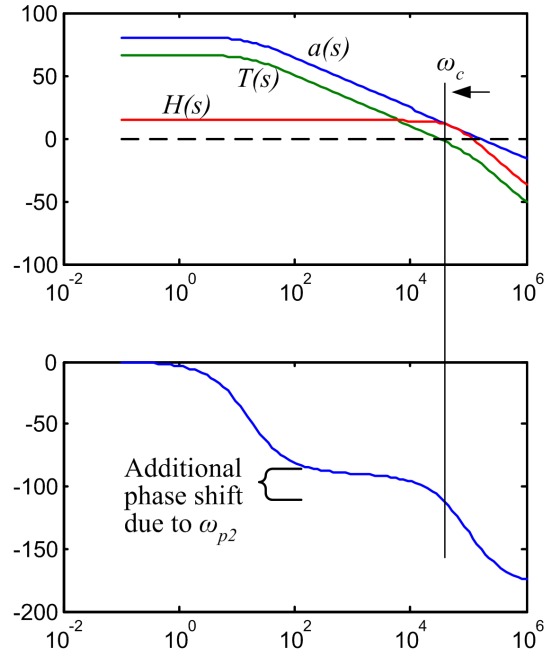


Figure 6: Circuit characteristics when  $\omega_{p2} \approx 3\omega_c$