The Nyquist criterion using Bode plots

Because of the simplicity of the graphical construction of the frequency response characteristics of a given transfer function the application of the Nyquist criterion is often more simple using Bode plots. The continuous change of the angle $\Delta\varphi_S$ of the vector from the critical point (-1,j0) to the locus of $G_0(j\omega)$ must be expressed by the amplitude and phase response of $G_0(j\omega)$. From Figure 5.7

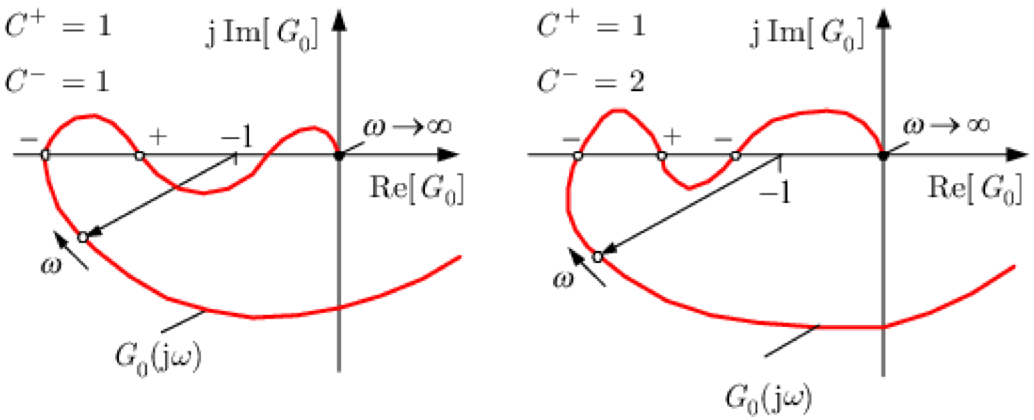


Figure 5.7: Positive (+) and negative (-) intersections of the locus $G_0(j\omega)$ with the real axis on the left-hand side of the critical point

it can be seen that this change of the angle is directly related to the count of intersections of the locus with the real axis on the left-hand side of the critical point between $(-\infty, -1)$. The Nyquist criterion can therefore also represented by the count of these intersections if the gain of the open loop is positive.

Regarding the intersections of the locus of $G_0(j\omega)$ with the real axis in the range $(-\infty, -1)$, the transfer from the upper to the lower half plane in the direction of increasing ω values are treated as *positive intersections* while the reverse transfer are *negative intersections* (Figure 5.7). The change of the angle is zero if the count of positive intersections S^+ is equal to the count of negative intersections S^-. The change of the angle $\Delta\varphi_S$ depends also on the number of positive and negative intersections and if the open loop does not have poles on the imaginary axis, the change of the angle is

$$\Delta\varphi_S = 2\pi(C^+ - C^-)\ .$$

In the case of an open loop containing an integrator, i.e. a single pole in the origin of the complex plane ($\mu = 1$), the locus starts for $\omega = 0$ at $\delta - j\infty$, where an additional $+\pi/2$ is added to the change of the angle. For proportional and integral behaviour of the open loop

$$\Delta\varphi_S = 2\pi(C^+ - C^-) + \mu\pi/2 \qquad \mu = 0, 1 \tag{5.16}$$

is valid. In principle this relation is also valid for $\mu = 2$, but the locus starts for $\omega = 0$ at $-\infty + j\delta$ (Figure 5.8), and this intersection would be counted

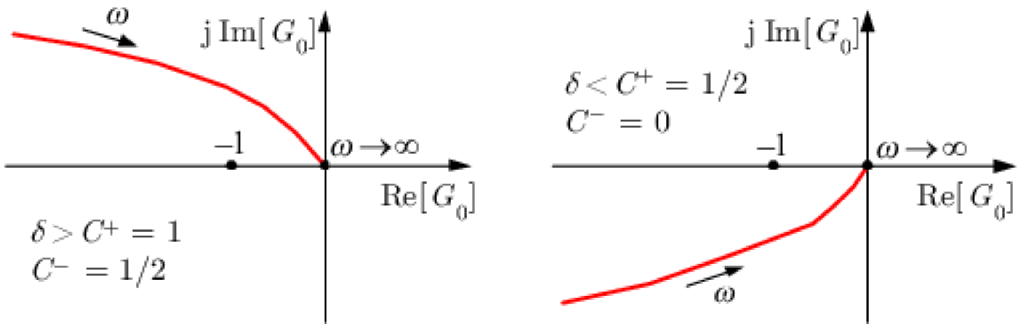


Figure 5.8: Count of the intersections on the left-hand side of the critical point for I_2 behaviour of the open loop

as a negative one if $\delta > 0$, i.e. if the locus for small ω is in the upper half plane of the real axis. But de facto there is for $\delta > 0$ (and accordingly $\delta < 0$) no intersection. This follows from the detailed investigation of the discontinuous change of the angle, which occurs at $\omega = 0$. As only a continuous change of the angle is taken into account and because of reason of symmetry the start of the locus at $\omega = 0$ is counted as a half intersection, positive for $\delta < 0$ and negative for $\delta > 0$, which is analogous to the definition given above (Figure 5.8). For continuous changes of the angle

$$\Delta\varphi_S = 2\pi(C^+ - C^-) \qquad (\mu = 2) \tag{5.17}$$

is valid. Comparing Eqs. (5.16) and (5.17), respectively, with Eq. (5.15) then the Nyquist criterion can be formulated as:

The open loop with the transfer function $G_0(s)$ has P poles in the left-half s plane and possibly a single ($\mu = 1$) or double pole ($\mu = 2$) at $s = 0$. If the locus of $G_0(j\omega)$ has C^+ positive and C^- negative intersections with the real axis to the left of the critical point, then the closed loop is only asymptotically stable, if

$$D^* = C^+ - C^- = \begin{cases} \dfrac{P}{2} & \text{for } \mu = 0,1 \\ \dfrac{P+1}{2} & \text{for } \mu = 2 \end{cases} \qquad (5.18)$$

is valid. For the special case, that the open loop is stable ($P = 0,\ \mu = 0$), the number of positive and negative intersections must be equal.

From this it follows that the difference of the number of positive and negative intersections in the case of $\mu = 0,1$ is an integer and for $\mu = 2$ not an integer. From this follows immediately, that for $\mu = 0,1$ the number P is even, for $\mu = 2$ the number $P+1$ is uneven and therefore in all cases P is an even number, such that the closed loop is asymptotically stable. This is only valid if $D^* \geq 1$.

The Nyquist criterion can now be transferred directly into the representation using frequency response characteristics. The magnitude response $A_0(\omega)_{\mathrm{dB}}$, which corresponds to the locus $G_0(j\omega)$, is always positive at the intersections of the locus with the real axis in the range of $(-\infty, -1)$. These points of intersection correspond to the crossings of the phase response $\varphi_0(\omega)$ with lines $\pm 180°$, $\pm 540°$ etc., i.e. a uneven multiple of 180 °. In the case of a positive intersection of the locus, the phase response at the $\pm(2k+1)\,180°$ lines crosses from below to top and reverse from top to below on a negative intersection as shown in Figure 5.9. In the following these crossings

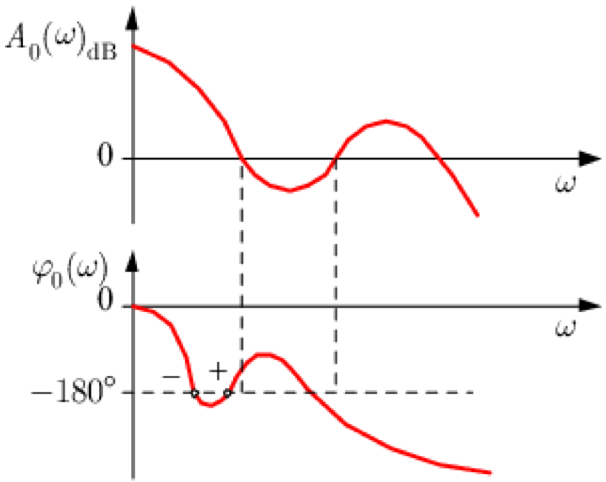


Figure 5.9: Frequency response characteristics of $G_0(j\omega) = A_0(\omega)\, e^{j\varphi_0(\omega)}$ and

definition of positive (+) and negative (-) crossings of the phase response $\varphi_0(\omega)$ with

the -180 ° line

will be defined as positive (+) and negative (-) crossings of the phase response $\varphi_0(\omega)$ over the particular $\pm(2k+1)\,180°$ lines, where $k = 0, 1, 2, \ldots$ may be valid. If the phase response starts at -180 ° this point is counted as a half crossing with the corresponding sign. Based on the discussions above the Nyquist criterion can be formulated in a form suitable for frequency response characteristics:

> The open loop with the transfer function $G_0(s)$ has P poles in the right-half s plane, and possibly a single or double pole at $s = 0$. C^+ are the number of positive and C^- of negative crossings of the phase response $\varphi_0(\omega)$ over the $\pm(2k+1)\,180°$ lines in the frequency range where $A_0(\omega)_{dB} > 0$ is valid. The closed loop is only asymptotically stable, if

$$D^* = C^+ - C^- = \begin{cases} \dfrac{P}{2} & \text{for } \mu = 0, 1 \\ \dfrac{P+1}{2} & \text{for } \mu = 2 \end{cases}$$

is valid. For the special case of an open-loop stable system ($P = 0,\ \mu = 0$)

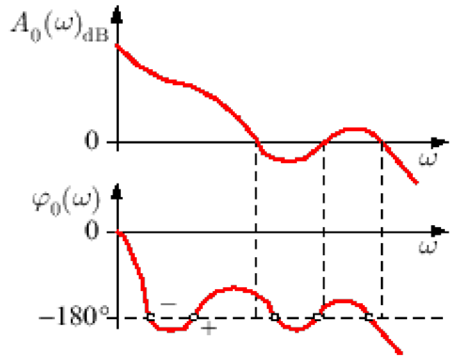
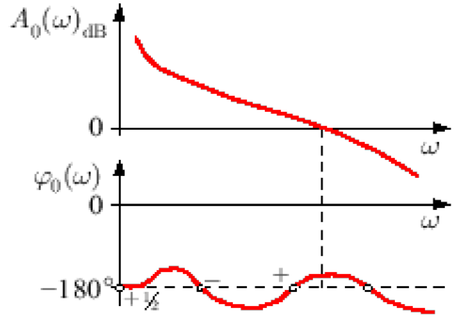
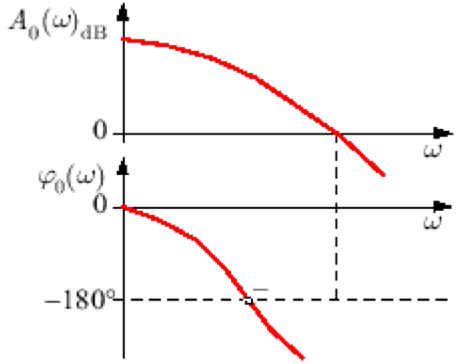
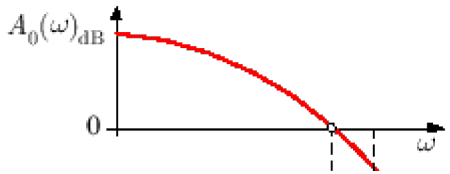
$$D^* = C^+ - C^- = 0$$

must be valid.

Table 5.1 shows some examples of the Nyquist criterion in the representation using frequency response characteristics.

Table 5.1: Examples of stability analysis using the Nyquist criterion with frequency response characteristics

Table 5.1: Examples of stability analysis using the Nyquist criterion with frequency response characteristics

No.	Bode Diagram	Stability Analysis
1		$\Rightarrow \left.\begin{array}{r} S^+ = 1 \\ S^- = 2 \\ D^* = -1 \\ P = 2 \end{array}\right\} \Rightarrow D^* \neq P/2$: unstable
2		$\Rightarrow \left.\begin{array}{r} S^+ = 3/2 \\ S^- = 1 \\ D^* = 1/2 \\ P = 0 \end{array}\right\} \Rightarrow D^* = \frac{P+1}{2}$: stable if 2 poles in the origin
3		$\Rightarrow \left.\begin{array}{r} S^+ = 0 \\ S^- = 1 \\ D^* = -1 \\ P = 0 \end{array}\right\} \Rightarrow D^* \neq P/2$: unstable
		$S^+ = 0$ $S^- = 0$

Finally the 'left-hand rule' will be given using Bode diagrams, because this version is for the most cases sufficient and simple to apply.

> The open loop has only poles in the left-half s plane with the exception of possibly one single or one multiple pole at $s = 0$ (P, I or I_2 behaviour). In this case the closed loop is only asymptotically stable, if $G_0(j\omega)$ has a phase of $\varphi_0 > -180°$ for the *crossover frequency* ω_C at $A_0(\omega_C)_{dB} = 0$.

This stability criterion offers the possibility of a practical assessment of the 'quality of stability' of a control loop. The larger the distance of the locus from the critical point the farther is the closed loop from the stability margin. As a measure of this distance the terms gain margin and phase margin are introduced according to Figure 5.10.

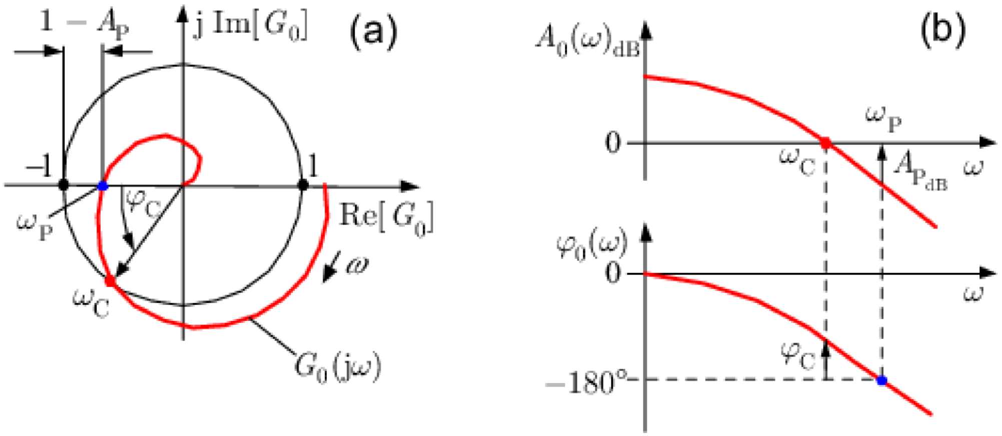


Figure 5.10: Phase and gain margin φ_C and A_P or $A_{P_{dB}}$, respectively, in the (a) Nyquist diagram and (b) Bode diagram

The *phase margin*

$$\varphi_C = 180° + \varphi_0(\omega_C) \tag{5.19}$$

is the distance of the phase response from the -180 ° line at the crossover frequency ω_C, i.e. at the crossing of the magnitude response with the 0 dB line ($|G_0| = 1$). The *gain margin*

(5.20)

$$A_{P_{dB}} = A_0(\omega_P)_{dB}$$

is the distance of the magnitude response from the 0 dB line at the phase of $\varphi_0 = -180°$.

A well damped control system should yield the following characteristics:

$$A_{P_{dB}} = \begin{cases} -12\,\mathrm{dB} & \text{to} & -20\,\mathrm{dB} & \text{for command response} \\ -3.5\,\mathrm{dB} & \text{to} & -9.5\,\mathrm{dB} & \text{for disturbance response} \end{cases}$$

$$\varphi_C = \begin{cases} 40° & \text{to} & 60° & \text{for command response} \\ 20° & \text{to} & 50° & \text{for disturbance response}\,. \end{cases}$$

The crossover frequency ω_C is a measure of the dynamical quality of the control loop. The higher ω_C the higher the bandwidth of the closed loop, and the faster the reaction on command inputs or disturbances. As the bandwidth that frequency is understood, at which the magnitude $A(\omega)$ of the closed-loop frequency response has fallen off approximately to zero.

Interactive Questions 5.2 Test your knowledge about gain and phase margin

Problem 5.1 Stability - three questions

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