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Research Article

Approximation Formula for Easy Calculation of Signal-to-Noise Ratio of Sigma-Delta Modulators

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The signal-to-noise ratio (SNR) is one of the most significant measures of performance of the sigma-delta modulators. An approximate formula for calculation of signal-to-noise ratio of an arbitrary sigma-delta modulator (SDM) has been proposed. Our approach for signal-to-noise ratio computation does not require modulator modeling and simulation. The proposed formula is compared with SNR calculations based on output bitstream obtained by simulations, and the reasons for small discrepancies are explained. The proposed approach is suitable for fast and precise signal-to-noise ratio computation. It is very useful in the modulator design stage, where multiple performance estimates are required.

1. Introduction

Sigma-delta modulation belongs to the group of pulse density modulation techniques, which exploits oversampling and sophisticated filter design in order to employ a low-bit quantizer with high effective resolution. Sigma-delta modulation is perhaps best understood by comparison with traditional pulse code modulation (PCM). A PCM converter typically samples an input signal at the Nyquist frequency and produces an N-bit representation of the original signal. This technique, however, requires quantization to 2N levels. Whether implemented using successive approximation registers, pipelined converters, or other techniques, high resolution is difficult to obtain in PCM conversion due to the need to accurately represent many quantization levels and the subsequent circuit complexity [1–7]. This obstacle is overcome with sigma-delta modulation, a form of pulse density modulation, which exploits oversampling and sophisticated filter design in order to employ a low-bit quantizer with high effective resolution. Due to this property, they are appropriate and are one of the main tools for analog-to-digital (A/D) signal conversion.

The most important parameter of all the A/D and D/A converters is the signal-to-noise ratio, which gives an estimate of modulator performance [6–8]. In the sigma-delta modulation, the SNR is one of the main parameters, but the calculation is always based on modulator output bitstream obtained by simulations. These simulations are usually time- and computational resource-consuming tasks. Some authors [4, 5, 8] have made an approximation of the expected SNR of ideal low-order SDM. On that basis, an extension can be made in order to approximate the SNR of a realistic SDM. For the engineering practice, where multiple parameter adjustments are made in order to obtain the best possible modulator, it is obviously better to use equations for fast SNR estimation. This is our motivations to derive a formula for approximate calculation of SNR for an arbitrary SDM.

In this paper, we consider common designs of sigma-delta modulators used for analog-to-digital conversion. The basic principle is the same for SDMs employed in D/A or sample rate conversion.

The paper is organized as follows. In the next section we present some basics of sigma-delta modulation and

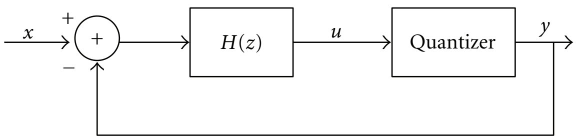

Figure 1: Basic structure of the sigma-delta modulator.

quantization noise. We also review the linear model, which assists in understanding filter design and noise shaping principles. In Section 3 we derive the SNR approximation formula. In Section 4 we give comparison of SNR results calculated with the derived approximation formula and SNR result calculations based on modulator output bitstream obtained by simulations. Finally, the conclusion remarks are given in the last section.

2. Sigma-Delta Modulation and Quantization Noise

The structure of a basic SDM is shown in Figure 1 and consists of a filter with transfer function $H(z)$ followed by a one-bit quantizer in a feedback loop. The system operates in discrete time.

The input to the loop is a discrete-time sequence $x(n) \in [-1, 1]$, which is to appear in quantized form at the output. The discrete-time sequence $u(n)$ is the output of the filter and the input to the quantizer. The output of the quantizer is the output bitstream of the modulator. The feedback loop acts in such a way that the quantization noise is shifted away from a certain frequency band. If an input signal within this frequency band is applied to the loop, most of the noise imposed by the quantization process lie, outside the frequency band of interest and can subsequently be filtered out, leaving a good approximation to the input signal [1–5].

The theory of quantization and the corresponding noise is well established (see [9] and references therein). The distance between 2 successive quantization levels is called the quantization step size, Q. For a quantizer with a specified number of bits covering the range from +1 to −1 there are 2^{bits} quantization levels, and the width of each quantization step is

$$Q = \frac{2}{(2^{\text{bits}} - 1)}. \tag{1}$$

The quantizer assigns each input sample $u(n)$ to the nearest quantization level, as shown on Figure 2. The quantization error is simply the difference between the input and output to the quantizer, $e_q = y(u) - u$, and is bounded by

$$-\frac{Q}{2} \leq e_q(n) \leq \frac{Q}{2}. \tag{2}$$

Since quantization is a highly nonlinear process, several assumptions are often made [7, 10].

(i) The error sequence, $e_q(n)$, is a stationary random process.

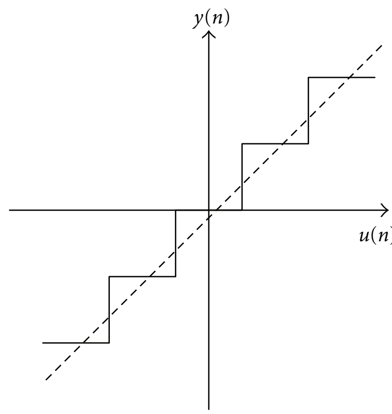

Figure 2: Quantization model.

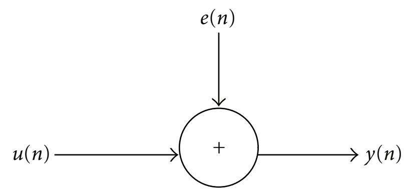

Figure 3: The quantizer under the linear model.

(ii) The error sequence is uncorrelated with itself and with the input sequence of the quantizer $u(n)$.

(iii) The probability density function of the error is uniform over the range of quantization error.

Such assumptions are a reasonable approximation for large amplitude time-varying input signals when number of quantizer bits is large and successive quantization error values are not highly correlated.

The assumption that the quantization error is uniformly distributed over a quantization step gives

$$P\left(e_q\right) = \begin{cases} \dfrac{1}{Q} & \left|e_q\right| \leq \dfrac{Q}{2}, \\ 0 & \left|e_q\right| > \dfrac{Q}{2}. \end{cases} \tag{3}$$

The quantization noise power is given by

$$\sigma_e^2 = \frac{1}{Q}\int_{-Q/2}^{Q/2} e_q^2 de_q = \frac{Q^2}{12} = \frac{1}{3(2^{\text{bits}} - 1)^2}. \tag{4}$$

Many authors [5, 10] propose σ_e^2 to be approximated to be

$$\sigma_e^2 \approx \frac{1}{3 \cdot 2^{2\text{bits}}}. \tag{5}$$

This quantization error is on the order of one least-significant-bit in amplitude and it is quite small compared to full-amplitude signals.

The average power of a sinusoidal signal of amplitude A, $x(t) = A\cos(2\pi t/T)$, is

$$\sigma_x^2 = \frac{1}{T}\int_0^T \left(A\cos\left(\frac{2\pi t}{T}\right)\right)^2 dt = \frac{A^2}{2}. \tag{6}$$

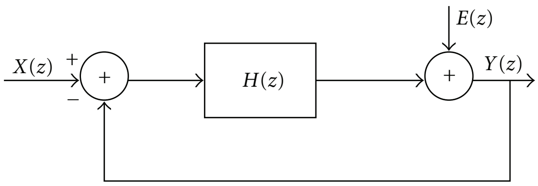

Figure 4: Representation of a sigma-delta modulator using the linear model.

Let us now assume that the signal is oversampled and the signal bandwidth is f_B. Rather than acquiring the signal at the Nyquist rate, $2f_B$, the actual sampling rate is $f_s = 2^{r+1} f_B$, with oversampling ratio $\mathrm{OSR} = 2^r = f_s/2f_B$. The quantization noise is spread over a larger frequency range, yet we are still primarily concerned with the noise below the Nyquist frequency.

Most of the noise power for SDM is located outside the signal band. The quantization noise power within the band of interest has decreased by a factor OSR. The signal power occurs over the signal band only, so it remains unchanged and is given by (6).

Many authors used the linear SDM model for analysis. The linear model is the most used model for modelling and analysis of the SDM effects. The essence of this model is the use of linearized model for the quantizer with quantizer substitution with error sequence $e(n)$, based on the assumption that the quantization error is uniformly distributed stationary random process, as shown in Figure 3.

The linear model contains two inputs: the input signal $X(z)$ and the quantization error $E(z)$. As the basic model shown in Figure 1, a filter is placed in front of the quantizer, known as the "loop filter" and the output of quantization is fed back and subtracted from the input signal, as shown in Figure 4.

This may be represented by transfer functions applied to both the input signal and the quantization noise. The Z-domain output may be represented as

$$Y(z) = \mathrm{STF}(z)X(z) + \mathrm{NTF}(z)E(z), \tag{7}$$

where the STF is the *signal transfer function* and the NTF is the *noise transfer function*. The input to the loop filter is $X(z) - E(z)$ so that $Y(z) = H(z)[X(z) - Y(z)] + E(z)$. Rearranging terms, we have [4, 5]

$$\mathrm{STF}(z) = \frac{H(z)}{1+H(z)}, \qquad \mathrm{NTF}(z) = \frac{1}{1+H(z)}. \tag{8}$$

3. Derivation of Approximation Formula

For our derivations we utilize the linear model of the SDM. The noise shaping in the SDM implies a nonconstant noise power in the baseband [4, 5],

$$\sigma_n^2 = \int_{-f_B}^{f_B} S_e^2(f)\,|\mathrm{NTF}(f)|^2 df, \tag{9}$$

where $S_e^2(f) = \sigma_e^2/f_s$ is the power spectral density of the unshaped quantization noise. The total noise power, σ_e^2, remains unchanged, but appropriate choice of the $\mathrm{NTF}(z)$ pushes the noise up to the high frequencies.

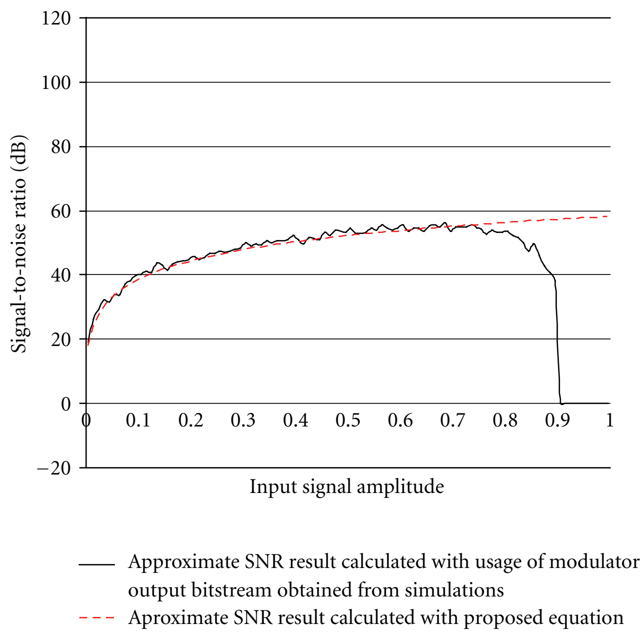

Figure 5: Estimates of the SNR as a function of input signal amplitude for a 1-bit, 3rd-order SDM with 64 times OSR for SDM with (18) NTF.

By definition, the SNR is calculated on basis of

$$\mathrm{SNR(dB)} = 10\log_{10}\frac{\sigma_x^2}{\sigma_n^2}. \tag{10}$$

From (4), (6), and (9), we can take the σ_e^2, σ_x^2, and σ_n^2 terms to substitute them in (10) to get the general formula for the SNR of any sigma-delta modulator:

$$\begin{aligned}\mathrm{SNR(dB)} &= 10\log_{10}\frac{\sigma_x^2}{\sigma_n^2}\\ &= 10\log_{10}\frac{A^2/2}{\int_{-f_B}^{f_B} S_e^2(f)\,|\mathrm{NTF}(f)|^2 df}\\ &= 10\log_{10}\frac{A^2/2}{(\sigma_e^2/f_s)\int_{-f_B}^{f_B}|\mathrm{NTF}(f)|^2 df}\\ &= 10\log_{10}\frac{A^2\cdot f_s}{2\sigma_e^2\int_{-f_B}^{f_B}|\mathrm{NTF}(f)|^2 df}.\end{aligned} \tag{11}$$

Applying approximation of σ_e^2 from (5) into (11) we get the formula

$$\begin{aligned}\mathrm{SNR(dB)} &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2\cdot f_s}{2\int_{-f_B}^{f_B}|\mathrm{NTF}(f)|^2 df}\\ &\approx 10\log_{10} 3\cdot 2^{2\mathrm{bits}} A^2 f_s\\ &\quad - 10\log_{10} 2\int_{-f_B}^{f_B}|\mathrm{NTF}(f)|^2 df.\end{aligned} \tag{12}$$

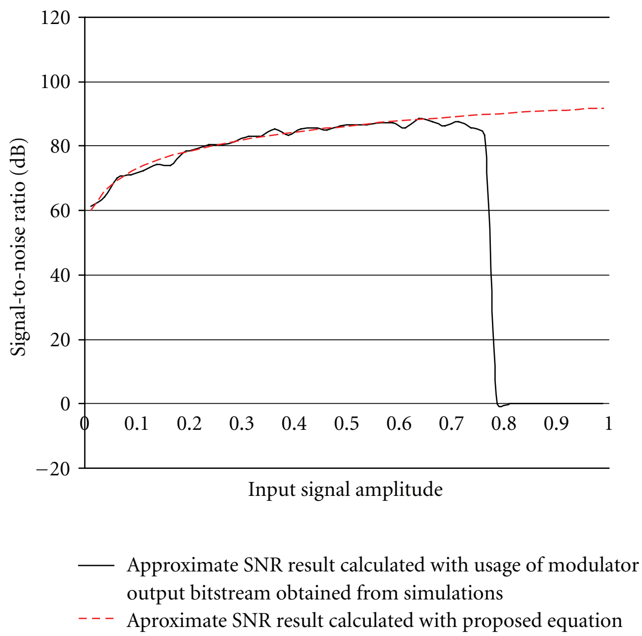

Figure 6: Estimates of the SNR as a function of input signal amplitude for a 1-bit, 3rd-order SDM with 64 times OSR for SDM with (19) NTF.

Using numerical integration, (12) may now be solved for a SDM with arbitrary noise transfer function, oversampling rate and bit length.

In order to verify the formula correctness, we made a comparison with other authors' formulae for approximation on maximal SNR, when using ideal NTFs. The archetypal ideal Nth-order SDMs are with NTF $= (1-z^{-1})^N$ and they are highly unstable for $N > 2$.

For modulator with first-order loopfilter with ideal NTF, the approximate SNR estimate is [5]

$$\mathrm{SNR(dB)} \approx 20\log_{10}A + 6.02\mathrm{bits} + 9.03r - 3.41. \tag{13}$$

In order to apply our approximate formula (12), the integral $\int_{-f_B}^{f_B} |\mathrm{NTF}(f)|^2 df$ must be solved for NTF $= (1-z^{-1})$.

Taking into account that

$$\begin{aligned} |\mathrm{NTF}(f)|^2 &= \left(1-e^{-j2\pi f/fs}\right)\cdot\left(1-e^{-j2\pi f/fs}\right) \\ &= 4\sin^2\left(\frac{\pi f}{fs}\right) \end{aligned} \tag{14}$$

and substituting (14) into $\int_{-f_B}^{f_B} |\mathrm{NTF}(f)|^2 df$, we get

$$\begin{aligned} \int_{-f_B}^{f_B} |\mathrm{NTF}(f)|^2 df &= \int_{-f_B}^{f_B} 4\sin^2\left(\frac{\pi f}{fs}\right) df \\ &= 4f_B - \frac{2fs}{\pi}\sin\left(\frac{2\pi f_B}{fs}\right) \\ &= \frac{2}{\mathrm{OSR}} - \frac{2}{\pi}\sin\left(\frac{\pi}{\mathrm{OSR}}\right). \end{aligned} \tag{15}$$

Finally, substituting (15) into (12), we obtain

$$\begin{aligned} \mathrm{SNR(dB)} &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2\cdot f_s}{2\int_{-f_B}^{f_B}|\mathrm{NTF}(f)|^2 df} \\ &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2\cdot f_s}{2[4f_B-(2f_s/\pi)\sin(2\pi f_B/f_s)]} \\ &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2}{2/f_s[4f_B-(2f_s/\pi)\sin(2\pi f_B/f_s)]} \\ &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2}{2(\pi^2/3\cdot \mathrm{OSR})} \\ &\approx 10\log_{10}\frac{3\cdot 2^{2\mathrm{bits}}\cdot A^2}{2(\pi^2/3\cdot 2^{3r})} \\ &\approx 10\log_{10}\frac{A^2\cdot 3^2\cdot 2^{2\mathrm{bits}}\cdot 2^{3r}}{2\pi^2}. \end{aligned} \tag{16}$$

Expanding (16) leads to the following result:

$$\begin{aligned} \mathrm{SNR(dB)} &\approx 10\log_{10}\frac{A^2\cdot 3^2\cdot 2^{2\mathrm{bits}}\cdot 2^{3r}}{2\pi^2} \\ &\approx 20\log_{10}A + 9.5424 \\ &\quad + 6.02\mathrm{bits} + 9.03r - 3.0103 - 9.9430 \\ &\approx 20\log_{10}A + 6.02\mathrm{bits} + 9.03r - 3.411. \end{aligned} \tag{17}$$

Equations (13) and (17) obtained by other authors for the ideal NTFs turn out to be a particular case of the proposed formula (12), which actually shows its correctness.

Although various approximations to this formula for ideal low-order designs have been provided in the literature ([4, 5] and references therein), to the authors' knowledge formulae for approximate SNR estimates have not yet been provided for practical, higher-order designs. The second term of (12) can be calculated very fast using numerical integration, and thus an approximate solution for the SDM SNR can be obtained.

4. Simulations and Comparison

In order to verify that the formula gives correct SNR results, we made a comparison with SNR calculations based on modulator output bitstream obtained by simulations. The first presented example includes two single-bit SDMs with realistic NTFs. For the SNR result approximation based on output bitstream obtained by simulations, we used MATLAB [11] and DSToolbox CalculateSNR function [12]. Figures 5 and 6 depict the SNR as a function of input signal amplitude for (12) and, based on output bitstream obtained by simulations SNR approximation estimates for

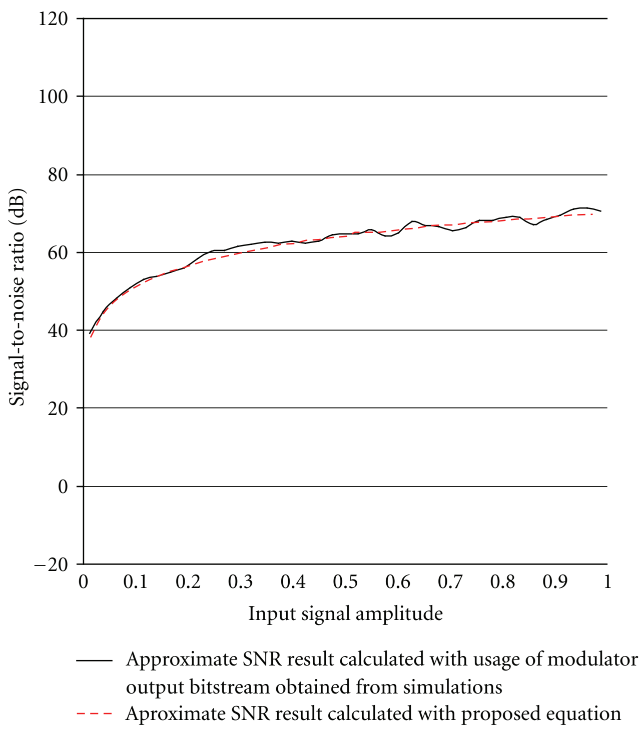

Figure 7: Estimates of the SNR as a function of input signal amplitude for a 3-bit, 3rd-order SDM with 64 times OSR for SDM with (18) NTF.

realistic SDMs. The compared realistic SDMs are of third-order, 1-bit lowpass implementation, intended for 64 times oversampling, with

$$\mathrm{NTF}(z) = \frac{z^3 - 3.049z^2 + 3.098949z - 1.049999}{z^3 - 2.2492z^2 + 1.747525z - 0.49783}, \quad (18)$$

$$\mathrm{NTF}(z) = \frac{z^3 - 2.999z^2 + 2.999z - 1}{z^3 - 2.1992z^2 + 1.6876z - 0.4441}. \quad (19)$$

The approximated SNR calculated from (12) is directly computed from the signal power and in-band noise power. In-band noise power is found from numerical integration of (12) using adaptive Simpson's quadrature, a standard technique. The approximated SNR calculated based on output bitstream obtained by simulations is done using 2^{18} bitstream datapoints. Sometimes on some amplitude steps the difference between the two graphs is as high as 0.5 dB.

The practical design that uses the NTF from (18) has an SNR almost 40 dB less than that of the ideal design, and simulation shows a dramatic drop in performance for input amplitude greater than 0.9. The sharp drop in the simulated SNR is also explained by instability. For sinusoidal inputs with magnitude above 0.9, the input to the quantizer grows exponentially and the output bitstream bears little relationship to the input signal (unstable).

The same type of modulator behaviour can be observed for SDM when using NTF from (19), with the exception of having higher performance and instability for lesser input signal amplitude levels. When the modulators are in the stable region of operation we observe a close match of the approximated SNR result between calculations based on (12) and calculations based on output bitstream obtained by simulations.

Figure 7 depicts the SNR as a function of input signal amplitude for both (12) and simulated SNR approximation estimates for (18) NTF realistic SDM third order, using 3-bit quantizer with 64 times OSR. The approximated SNR calculated based on output bitstream obtained by simulations for this example is done using 2^{17} bitstream datapoints. In this example the modulator is stable for the whole range

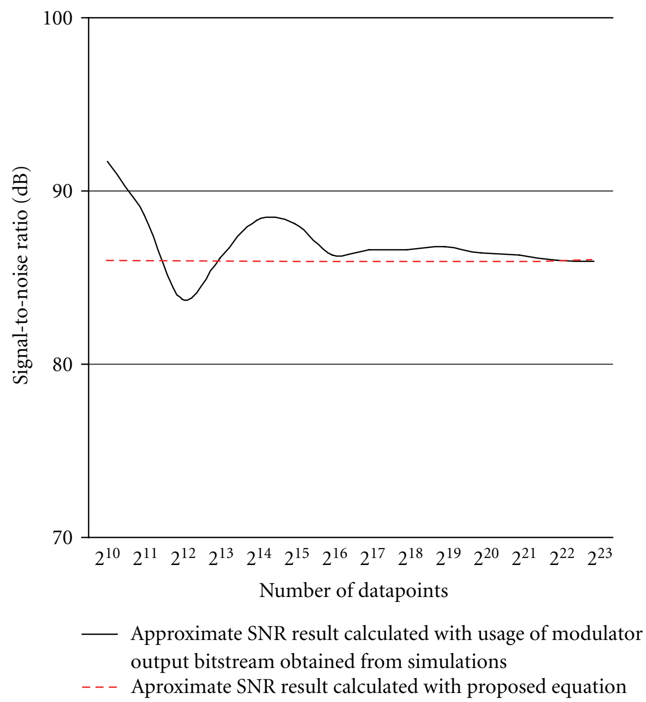

Figure 8: SNR result convergence rate of the DSToolbox function, according to the number of bitstream datapoints used.

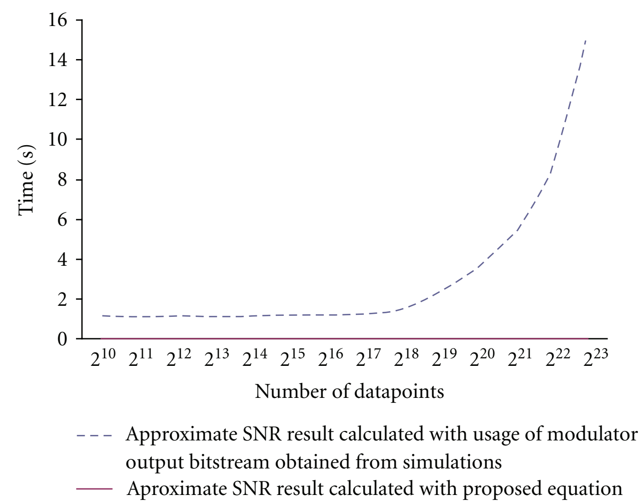

Figure 9: SNR result computational time passed for certain number of datpoints for simulations used.

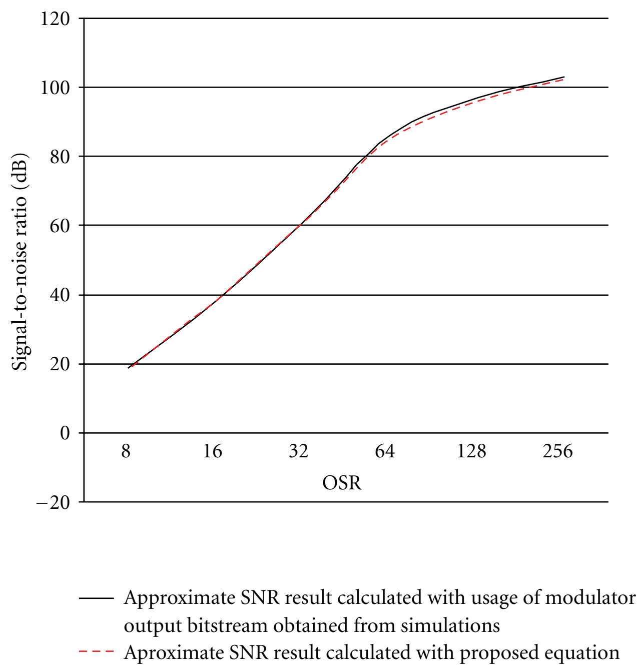

Figure 10: Estimates of the SNR as a function of OSR for a 1-bit, 3rd-order SDM (19).

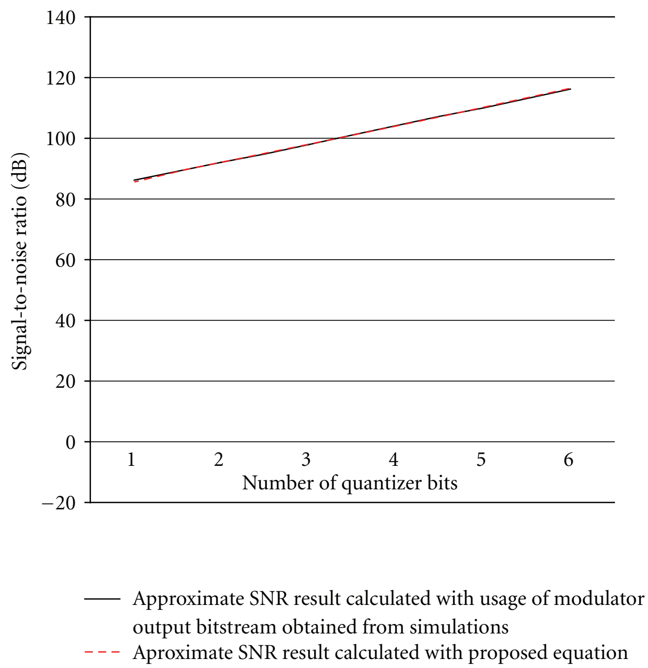

Figure 11: Estimates of the SNR as a function of number of bits for a 3rd-order SDM (19) with 64 times OSR.

of input signal amplitude, thanks to the higher bit quantizer. For this higher bit quantizer example, we have again, as for the single bit quantizer examples, a close relation between approximate SNR result calculation based on (12) and SNR calculation based on modulator output bitstream obtained by simulations.

The most important advantage of SNR calculation with (12) is the lack of a need to produce SDM model simulations. In order to obtain a really precise SNR calculation that is based on output bitstream obtained by simulations, first we must generate simulation output data from the model and then calculate the SNR. When using simulated data, a high number of data points must be used for accurate SNR estimation. For example, we used the CalculateSNR function that is part of the MATLAB DSToolbox [12] in order to verify the SNR result convergence when using higher number of datapoints in comparison with result from (12). An example for the SNR result convergence as a function of the number of data points for single-bit quantizer with third-order NTF (19) is shown in Figure 8. In this example the oversampling ratio is 64 and amplitude is $A = 0.5$.

(12) based SNR calculation is not dependent on the number of datapoints as the approximated from simulation SNR calculation is. The discrepancies of the calculated via simulation SNR exist thanks to the used fast Fourier transform. Because the quantization noise is not with perfect uniform distribution for low-order quantizers [13] sometimes the predicted result with formula (9) is not that precise. The regular SNR result deviation between proposed (12) calculation and calculation based on output bitstream obtained by simulations is usually no more than 0.3 dB, assuming the modulator is stable. We made our own function for SNR calculation based on modulator output bitstream obtained by simulations in order to verify the result we get from the DSToolbox function, and we can state safely, that for both of the implementations it is preferable to take at least 2^{23} modulator output bitstream simulation datapoints for SNR calculations. The SNR calculation based on output bitstream obtained by simulations does not fully converge until at least 2^{23} data points are used. At two million output bitstream datapoints taken from simulations and used for the SNR calculation, the formula derived results and the results, calculated by simulations are matched.

In Figure 9, the computational time taken on Intel Core2Duo T8300 processor to obtain the output bitstream from simulations for a certain number of datapoints including SNR calculation is presented. This result is obtained from simulating SDMs with DSToolbox simulateDSM function [12] as author-built SDM models are simulated for larger periods of time. Computational time needed to obtain SNR result with (12) is only a minimal fraction of the second. For comparison, SNR calculations with 2^{23} modulator output bitstream simulation datapoints at least 15 second per SNR result estimation is required. In the engineering practice, multiple parameter adjustments are made in the modulator design process. In these cases multiple modulator simulations are required on every design step in order to verify the modulator performance. For this, cases where multiple simulations are needed to get SNR result estimation, SNR calculation with (12) can be a viable tool, speeding the design process. This fast approach for the SNR calculation brings the possibility to design SDMs loopfilters with usage of standard optimization techniques. Finally, in Figures 10 and 11 are presented results for different values of the OSR and the quantizer bits number. Both curves in Figure 11 are almost indistinguishable.

5. Conclusion

In this paper, we derive approximate formula for calculating the signal-to-noise ratio of sigma-delta modulators and compare the result with SNR computation based on modulator output bitstream obtained by simulations.

The calculation of the formula is very fast, because it only needs a simple numerical integration. The novelty of this approach is that in order to have a precise SNR calculation, there is no need of modulator simulations, thus leading to instantaneous SNR estimate. The main advantage of this approach is that it removes the need for computationally intensive simulation to estimate performance (output bitstream), and it can also be used to validate simulations. Furthermore, with the usage of this fast approach for the SNR calculation, it may be possible to design SDMs when using standard optimization techniques.

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