Decoupling Theorem: For any lossless reciprocal three-port network, one port can be terminated in a reactance so that the other two ports are decoupled.

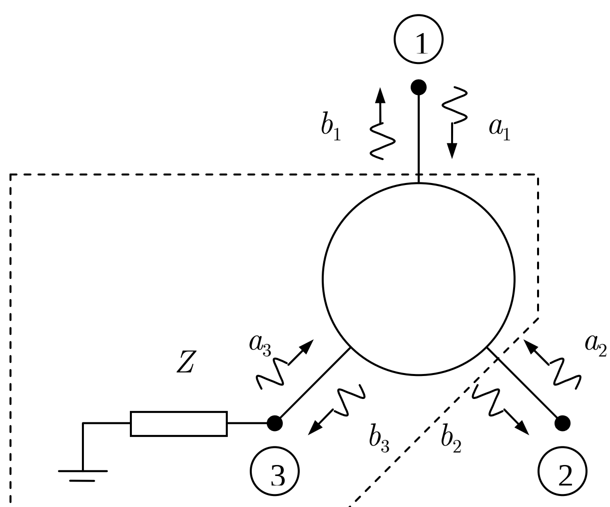
Proof:

For the shown 3-port network with port 3 terminated in a load, let a source be attached to port 1 and a matched load be attached to port 2 ($a_2 = 0$), thus:

$$\begin{aligned} b_1 &= s_{11}a_1 + s_{13}a_3 \\ b_2 &= s_{12}a_1 + s_{23}a_3 \\ b_3 &= s_{13}a_1 + s_{33}a_3 \end{aligned} \tag{1}$$

Notice that the fact the network is reciprocal was used in (1). Since the network is also lossless, then we have

$$\begin{array}{l|l} |s_{11}|^2 + |s_{12}|^2 + |s_{13}|^2 = 1 & \quad s_{11}s_{12}^* + s_{12}s_{22}^* + s_{13}s_{23}^* = 0 \\ |s_{12}|^2 + |s_{22}|^2 + |s_{23}|^2 = 1 & \quad s_{12}s_{13}^* + s_{22}s_{23}^* + s_{23}s_{33}^* = 0 \\ |s_{13}|^2 + |s_{23}|^2 + |s_{33}|^2 = 1 & \quad s_{11}s_{13}^* + s_{12}s_{23}^* + s_{13}s_{33}^* = 0 \end{array} \tag{2}$$



and at port 3, we have

$$a_3 = \Gamma b_3, \qquad \Gamma = \frac{Z - Z_0}{Z + Z_0} \tag{3}$$

From (1) and (3)

$$
\begin{aligned}
b_2 &= s_{12}a_1 + s_{23}a_3 = s_{12}a_1 + s_{23}\Gamma b_3 = s_{12}a_1 + s_{23}\Gamma\left(s_{13}a_1 + s_{33}a_3\right) \\
&= \left(s_{12} + s_{23}s_{13}\Gamma\right)a_1 + s_{23}s_{33}\Gamma a_3 = \left(s_{12} + s_{23}s_{13}\Gamma\right)a_1 + s_{23}s_{33}\Gamma\frac{b_2 - s_{12}a_1}{s_{23}} \\
&= \left(s_{12} + s_{23}s_{13}\Gamma - s_{33}s_{12}\Gamma\right)a_1 + s_{33}\Gamma b_2
\end{aligned}
$$

$$
\left.\frac{b_2}{a_1}\right|_{a_2=0} = \frac{\left(s_{12} + s_{23}s_{13}\Gamma - s_{33}s_{12}\Gamma\right)}{1 - s_{33}\Gamma} = \frac{\left(1 - s_{33}\Gamma\right)s_{12} + s_{23}s_{13}\Gamma}{1 - s_{33}\Gamma}
$$

$$
s'_{21} \equiv \left.\frac{b_2}{a_1}\right|_{a_2=0} = s_{12} + \frac{\Gamma s_{13}s_{23}}{1 - \Gamma s_{33}} \tag{4}
$$

where s'_{21} is the transmission coefficient from port 1 to 2 in the new two-port network (enclosed by the dotted line in the figure). For s'_{21} to be equal to 0, Γ should satisfy the condition

$$
\Gamma = \frac{s_{12}}{s_{12}s_{33} - s_{13}s_{23}} \tag{5}
$$

Then

$$
\begin{aligned}
\left|\Gamma\right|^2 = \Gamma\Gamma^* &= \frac{\left|s_{12}\right|^2}{\left|s_{12}\right|^2\left|s_{33}\right|^2 + \left|s_{13}\right|^2\left|s_{23}\right|^2 - s_{12}s_{33}\overset{*}{s}_{13}\overset{*}{s}_{23} - \overset{*}{s}_{12}\overset{*}{s}_{33}s_{13}s_{23}} \\
&= \frac{\left|s_{12}\right|^2}{\left\{\begin{array}{l} \left|s_{12}\right|^2\left(1 \cancel{-\left|s_{13}\right|^2} \cancel{-\left|s_{23}\right|^2}\right) + \left|s_{13}\right|^2\left|s_{23}\right|^2 \\ \qquad -s_{12}\overset{*}{s}_{23}\left(\cancel{-\overset{*}{s}_{12}s_{23}} - \overset{*}{s}_{11}s_{13}\right) - \overset{*}{s}_{12}s_{13}\left(-s_{22}\overset{*}{s}_{23} \cancel{-s_{12}\overset{*}{s}_{13}}\right) \end{array}\right\}} \\
&= \frac{\left|s_{12}\right|^2}{\left|s_{12}\right|^2 + s_{13}\overset{*}{s}_{13}s_{23}\overset{*}{s}_{23} + s_{12}\overset{*}{s}_{23}\overset{*}{s}_{11}s_{13} + \overset{*}{s}_{12}s_{13}s_{22}\overset{*}{s}_{23}} \\
&= \frac{\left|s_{12}\right|^2}{\left|s_{12}\right|^2 + s_{13}\overset{*}{s}_{23}\underbrace{\left(\overset{*}{s}_{13}s_{23} + s_{12}\overset{*}{s}_{11} + \overset{*}{s}_{12}s_{22}\right)}_{=0}} = 1
\end{aligned} \tag{6}
$$

Since $\left|\Gamma\right| = 1$, then $Z = jX$.