Network Parameters

Impedance and Admittance matrices

For n ports network we can relate the voltages and currents by impedance and admittance matrices

Impedance matrix

$$\begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ V_n \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{21} & . & . & Z_{n1} \\ Z_{12} & Z_{22} & . & . & Z_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ Z_{1n} & Z_{2n} & . & . & Z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ . \\ . \\ I_n \end{bmatrix}$$

Admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \\ . \\ . \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{21} & . & . & Y_{n1} \\ Y_{12} & Y_{22} & . & . & Y_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ Y_{1n} & Y_{2n} & . & . & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ V_n \end{bmatrix}$$

where $[Y] = [Z]^{-1}$

Reciprocal and Lossless Networks

Reciprocal networks usually contain nonreciprocal media such as ferrites or plasma, or active devices. We can show that the impedance and admittance matrices are symmetrical, so that.

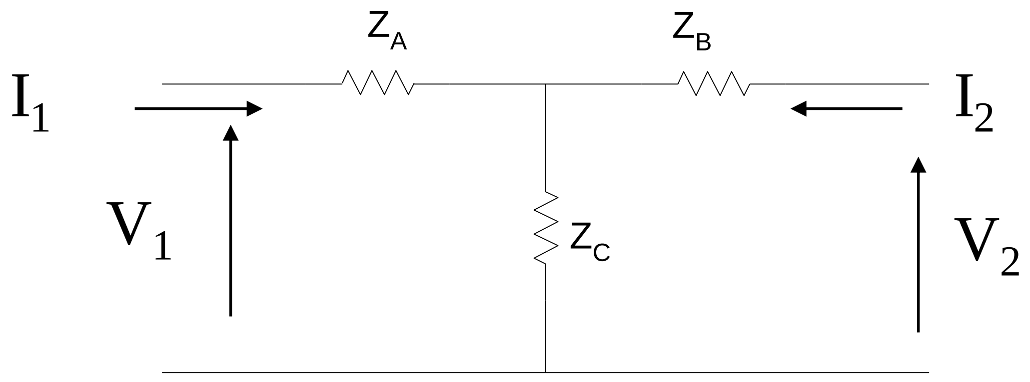
$$Z_{ij} = Z_{ji} \text{ or } Y_{ij} = Y_{ji}$$

Lossless networks can be shown that Z_{ij} or Y_{ij} are imaginary

Refer to text book Pozar pg193-195

Example

Find the Z parameters of the two-port T –network as shown below



Solution

Port 2 open-circuited

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_A + Z_C$$

Port 1 open-circuited

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{V_2}{I_2}\frac{Z_C}{Z_B+Z_C} = \left(Z_B+Z_C\right)\frac{Z_C}{Z_B+Z_C} = Z_C$$

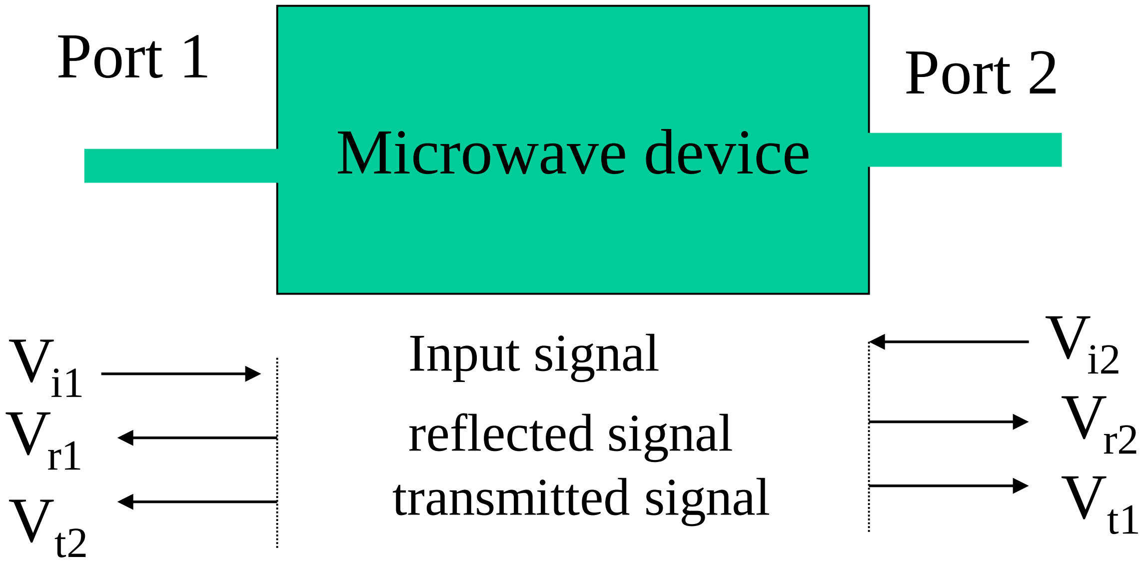
Similarly we can show that

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_C$$

This is an example of reciprocal network!!

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_B + Z_C$$

S-parameters



Transmission and reflection coefficients

$$\tau = \frac{V_t}{V_i} \qquad \rho = \frac{V_r}{V_i}$$

S-parameters

Voltage of traveling wave away from port 1 is

$$V_{b1} = \frac{V_{r1}}{V_{i1}} V_{i1} + \frac{V_{t2}}{V_{i2}} V_{i2}$$

Voltage of Reflected wave From port 1

Voltage of Transmitted wave From port 2

Voltage of transmitted wave away from port 2 is

$$V_{b2} = \frac{V_{t1}}{V_{i1}} V_{i1} + \frac{V_{r2}}{V_{i2}} V_{i2}$$

Let $V_{b1}= b_1$, $V_{i1}=a_1$, $V_{i2}=a_2$, $\rho_1 = \frac{V_{r1}}{V_{i1}}$, $\tau_{12} = \frac{V_{t2}}{V_{i2}}$, $\tau_{21} = \frac{V_{t1}}{V_{i1}}$

and $\rho_2 = \frac{V_{r2}}{V_{i2}}$

Then we can rewrite

S-parameters

Hence

$$b_1 = \rho_1 \, a_1 + \tau_{12} a_2$$

$$b_2 = \tau_{21} \, a_1 + \rho_{\,2} a_2$$

In matrix form

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \rho_1 & \tau_{12} \\ \tau_{21} & \rho_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

S-matrix

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

- $S_{11}$and S_{22} are a measure of reflected signal at port 1 and port 2 respectively
- S_{21} is a measure of gain or loss of a signal from port 1 to port 2.
- S12 ia a measure of gain or loss of a signal from port 2 to port 1.

Logarithmic form

S_{11}=20 log(ρ_1)

S_{22}=20 log(ρ_2)

S_{12}=20 log(τ_{12})

S_{21}=20 log(τ_{21})

S-parameters

$$S_{11} = \left.\frac{V_{r1}}{V_{i1}}\right|_{V_{r2}=0}$$

$$S_{12} = \left.\frac{V_{t2}}{V_{i2}}\right|_{V_{r2}=0}$$

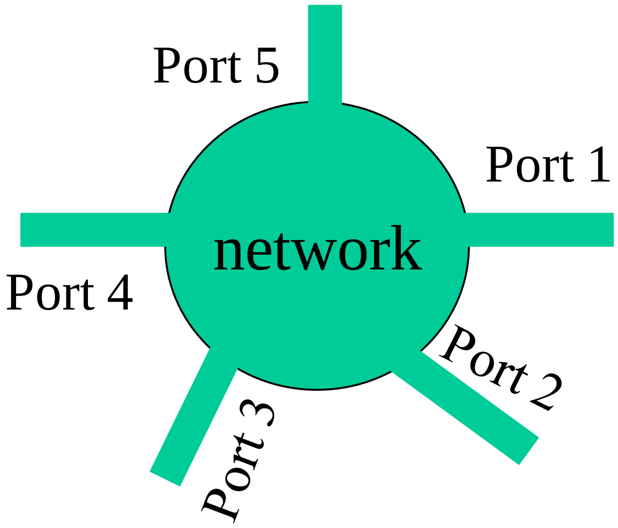
V_{r2}=0 means port 2 is matched

$$S_{21} = \left.\frac{V_{t1}}{V_{i1}}\right|_{V_{r1}=0}$$

$$S_{22} = \left.\frac{V_{r2}}{V_{i2}}\right|_{V_{r1}=0}$$

V_{r1}=0 means port 1 is matched

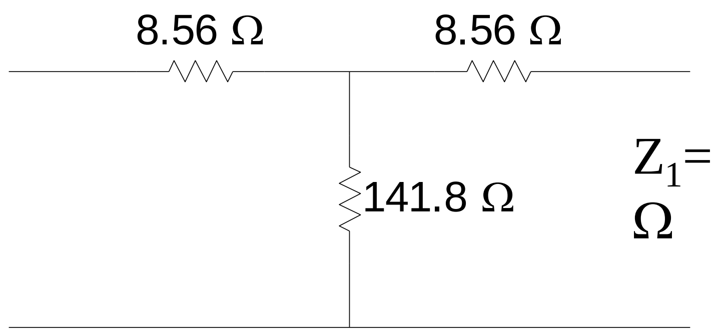
Multi-port network



$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

Example

Below is a matched 3 dB attenuator. Find the S-parameter of the circuit.



$Z_1 = Z_2 = 8.56\ \Omega$ and $Z_3 = 141.8\ \Omega$

Solution

$$S_{11} = \left.\frac{V_{r1}}{V_{i1}}\right|_{V_{r2}=0} = \rho = \frac{Z_{in} - Z_o}{Z_{in} + Z_o}$$

By assuming the output port is terminated by $Z_o = 50\ \Omega$, then

$$Z_{in} = Z_1 + \left[Z_3 // (Z_2 + Z_o) \right]$$

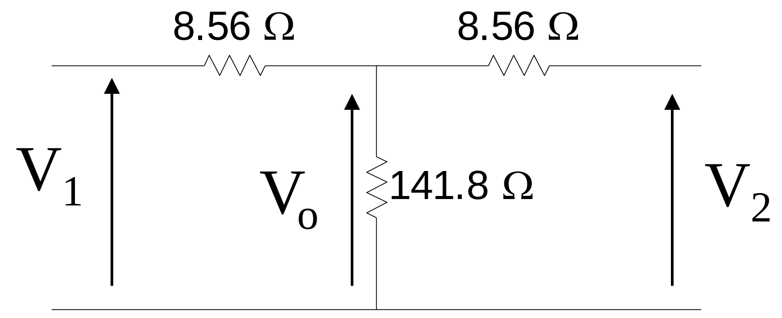
$$= 8.56 + \left[141.8(8.56 + 50) / (141.8 + 8.56 + 50) \right] = 50\ \Omega$$

$$S_{11} = \frac{50 - 50}{50 + 50} = 0$$

Because of symmetry , then S_{22}=0

Continue

$$S_{21} = \frac{V_{t2}}{V_{i2}}\bigg|_{V_{r2}=0}$$



From the fact that $S_{11}=S_{22}=0$, we know that $V_{r1}=0$ when port 2 is matched, and that $V_{i2}=0$. Therefore $V_{i1}= V_1$ and $V_{t2}=V2$

$$V_{t2} = V_2 = V_1\left(\frac{Z_2 // Z_3}{Z_2 // Z_3 + Z_1}\right)\left(\frac{Z_o}{Z_3 + Z_o}\right) = V_o\left(\frac{Z_o}{Z_3 + Z_o}\right)$$

$$= V_1\left(\frac{41.44}{41.44 + 8.56}\right)\left(\frac{50}{50 + 8.56}\right) = 0.707V_1$$

Therefore $S_{12} = S_{21} = 0.707$

$$[S] = \begin{bmatrix} 0 & 0.707 \\ 0.707 & 0 \end{bmatrix}$$

Lossless network

For lossless n-network , total input power = total output power. Thus

$$\sum_{i=1}^{n} a_i a_i^* = \sum_{i=1}^{n} b_i b_i^*$$

Where a and b are the amplitude of the signal

Putting in matrix form $a^t a^* = b^t b^*$

$=a^t S^t S^* a^*$

Note that $b^t=a^tS^t$ and $b^*=S^*a^*$

Thus $a^t (I - S^t S^*)a^* = 0$ This implies that $S^t S^* = I$

Called unitary matrix

In summation form

$$\sum_{k=1}^{n} S_{ki} S_{kj}^* = \left.\begin{matrix} 1 & for\, i = j \\ 0 & for\, i \neq j \end{matrix}\right\}$$

Conversion of Z to S and S to Z

$$[S] = ([Z] + [U])^{-1}([Z] - [U])$$

$$[Z] = ([U] - [S])^{-1}([U] + [S])$$

where

$$[U] = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & . & . & . \\ . & . & 1 & . \\ 0 & . & . & 1 \end{bmatrix}$$

Reciprocal and symmetrical network

Since the [U] is diagonal , thus $[U] = [U]^t$

For reciprocal network

$$[Z] = [Z]^t$$

Since [Z] is symmetry

Thus it can be shown that

$$[S] = [S]^t$$

Example

A certain two-port network is measured and the following scattering matrix is obtained:

$$[S] = \begin{bmatrix} 0.1\angle 0^o & 0.8\angle 90^o \\ 0.8\angle 90^o & 0.2\angle 0^o \end{bmatrix}$$

From the data , determine whether the network is reciprocal or lossless. If a short circuit is placed on port 2, what will be the resulting return loss at port 1?

Solution

Since [S] is symmetry, the network is reciprocal. To be lossless, the S parameters must satisfy

$$\sum_{k=1}^{n} S_{ki} S_{kj}^{*} = \left. \begin{matrix} 1 & for\, i = j \\ 0 & for\, i \neq j \end{matrix} \right\}$$

For i=j

$|S_{11}|^2 + |S_{12}|^2 = (0.1)^2 + (0.8)^2 = 0.65$

Since the summation is not equal to 1, thus it is not a lossless network.

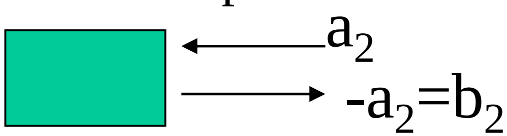
continue

Reflected power at port 1 when port 2 is shorted can be calculated as follow and the fact that $a_2 = -b_2$ for port 2 being short circuited, thus

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 - S_{12}b_2 \qquad (1)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 - S_{22}b_2 \qquad (2)$$

Short at port 2



From (2) we have

$$b_2 = \frac{S_{21}}{1+S_{22}} a_1 \qquad (3)$$

Dividing (1) by a_1 and substitute the result in (3) ,we have

$$\rho = \frac{b_1}{a_1} = S_{11} - S_{12}\frac{b_2}{a_1} = S_{11} - \frac{S_{12}S_{21}}{1+S_{22}} = 0.1 - \frac{(j0.8)(j0.8)}{1+0.2} = 0.633$$

Return loss $\quad -20\log\rho = -20\log(0.633) = 3.97\,dB$

ABCD parameters

Voltages and currents in a general circuit

$$I_2 \propto V_2 - V_1 \qquad V_2 \propto I_1 - I_2$$

This can be written as

$$V_1 \propto V_2 - I_2 \qquad I_1 \propto V_2 + I_2$$

Or

$$V_1 = AV_2 - BI_2 \qquad I_1 = CV_2 - DI_2$$

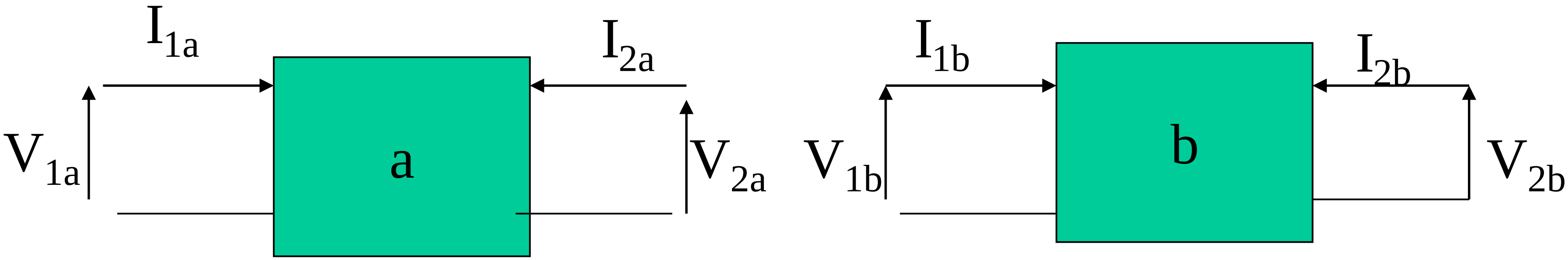
A –ve sign is included in the definition of D

In matrix form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Given V_1 and I_1, V_2 and I_2 can be determined if ABDC matrix is known.

Cascaded network



$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} \qquad \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

However $V_{2a}=V_{1b}$ and $-I_{2a}=I_{1b}$ then

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

The main use of ABCD matrices are for chaining circuit elements together

Or just convert to one matrix

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

Where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

Determination of ABCD parameters

$$V_1 = AV_2 - BI_2 \qquad I_1 = CV_2 - DI_2$$

Because A is independent of B, to determine A put I_2 equal to zero and determine the voltage gain V1/V2=A of the circuit. In this case port 2 must be open circuit.

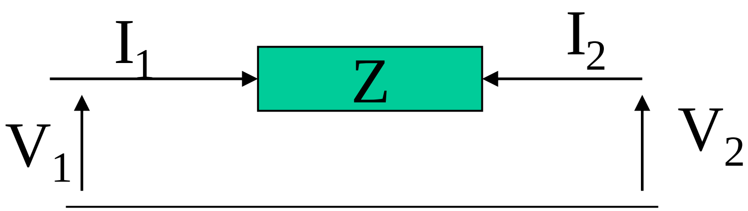
$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

ABCD matrix for series impedance



$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

$V_1 = V_2$ hence A=1

$V_1 = -I_2 Z$ hence B= Z

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

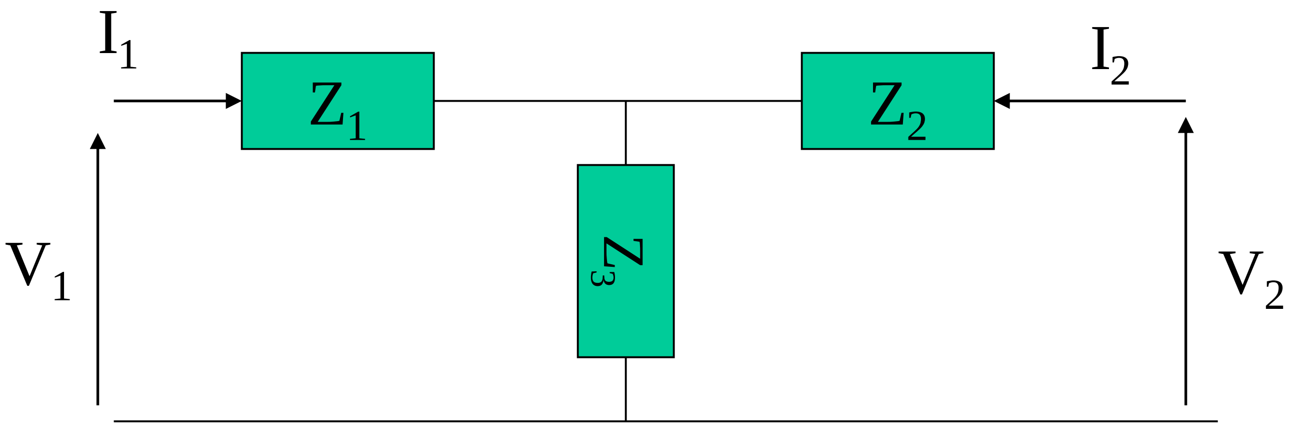
$I_1 = -I_2 = 0$ hence C= 0

$I_1 = -I_2$ hence D= 1

The full ABCD matrix can be written

$$\begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$$

ABCD for T impedance network



$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

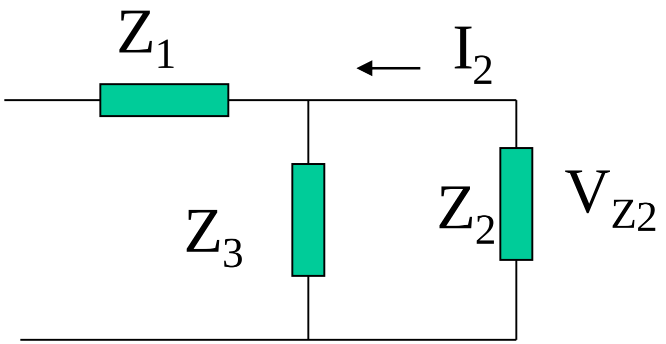
then $$V_2 = \frac{Z_3}{Z_1 + Z_3} V_1$$

therefore

$$A = \frac{V_1}{V_2} = \frac{Z_1 + Z_3}{Z_3} = 1 + \frac{Z_1}{Z_3}$$

Continue

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit



Solving for voltage in Z_2

$$V_{Z_2} = \frac{\dfrac{Z_2 Z_3}{Z_2 + Z_3}}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}} V_1$$

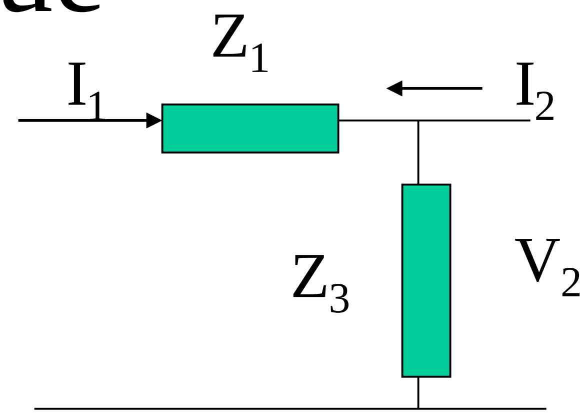
But

$$V_{Z_2} = -I_2 Z_2$$

Hence

$$B = \frac{V_1}{-I_2} = Z_2 + Z_1 + \frac{Z_1 Z_2}{Z_3}$$

Continue



$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

Analysis

$$-I_2 = I_1$$

$$V_2 = -I_2 Z_3 = I_1 Z_3$$

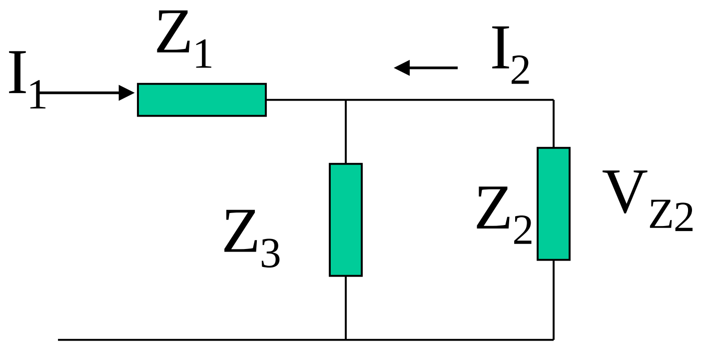
Therefore

$$C = \frac{I_1}{V_2} = \frac{1}{Z_3}$$

Continue

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$

for port 2 short circuit



I_1 is divided into Z_2 and Z_3, thus

$$I_2 = \frac{-Z_3}{Z_2 + Z_3} I_1$$

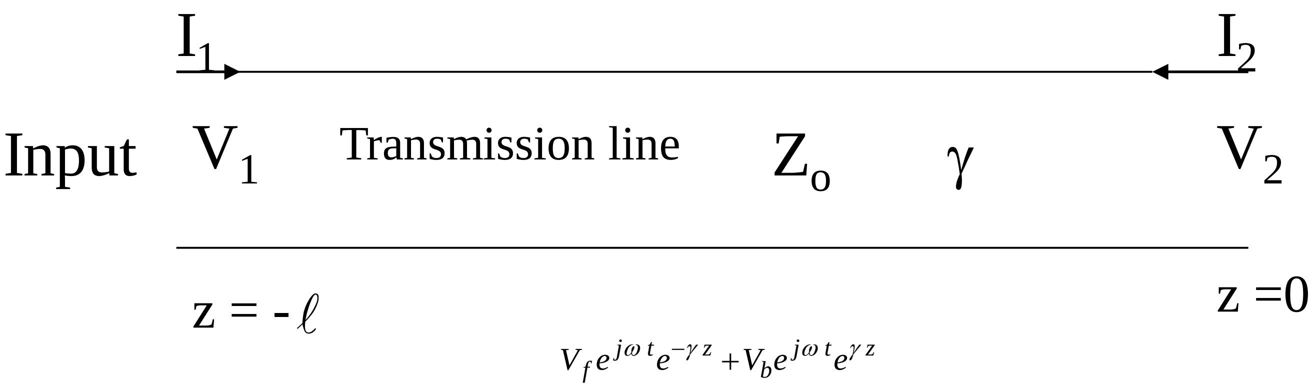
Hence

$$D = \frac{I_1}{-I_2} = 1 + \frac{Z_2}{Z_3}$$

Full matrix

$$\begin{bmatrix} 1 + \frac{Z_1}{Z_2} & Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3} \\ \frac{1}{Z_3} & 1 + \frac{Z_2}{Z_3} \end{bmatrix}$$

ABCD for transmission line



For transmission line

$$V(z) = V_f e^{j\omega t} e^{-\gamma z} + V_b e^{j\omega t} e^{\gamma z}$$

$$Z_o = \frac{V_f}{I_f} = \frac{V_b}{I_b}$$

$$I(z) = \frac{1}{Z_o}\left(V_f e^{j\omega t} e^{-\gamma z} - V_b e^{j\omega t} e^{\gamma z}\right)$$

f and b represent forward and backward propagation voltage and current Amplitudes. The time varying term can be dropped in further analysis.

continue

At the input z = - ℓ

$$V_1 = V(-\ell) = V_f e^{+\gamma \ell} + V_b e^{-\gamma \ell} \quad (1)$$

$$I_1 = I(-\ell) = \frac{1}{Z_o}\left(V_f e^{\gamma \ell} - V_b e^{-\gamma \ell}\right) \quad (2)$$

At the output z = 0

$$V_2 = V(0) = V_f + V_b \quad (3)$$

$$I_2 = I(0) = \frac{1}{Z_o}\left(V_f - V_b\right) \quad (4)$$

Now find A,B,C and D using the above 4 equations

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

For I_2 =0 Eq.(4) gives V_f= V_b=V_o giving

continue

From Eq. (1) and (3) we have

$$A = \frac{V_o(e^{\gamma\ell} + e^{-\gamma\ell})}{2V_o} = \cosh(\gamma\ell)$$

Note that

$$\cosh(x) = \frac{(e^{x} + e^{-x})}{2}$$

$$\sinh(x) = \frac{(e^{x} - e^{-x})}{2}$$

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

For $V_2 = 0$, Eq. (3) implies $-V_f = V_b = V_o$. From Eq. (1) and (4) we have

$$B = \frac{Z_o V_o(e^{\gamma\ell} - e^{-\gamma\ell})}{2V_o} = Z_o \sinh(\gamma\ell)$$

continue

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$ for port 2 open circuit

For $I_2=0$, Eq. (4) implies $V_f = V_b = V_o$. From Eq.(2) and (3) we have

$$C = \frac{V_o(e^{\gamma\ell} - e^{-\gamma\ell})}{2V_oZ_o} = \frac{\sinh(\gamma\ell)}{Z_o}$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$ for port 2 short circuit

For $V_2=0$, Eq. (3) implies $V_f = -V_b = V_o$. From Eq.(2) and (4) we have

$$D = \frac{-Z_oV_o(e^{\gamma\ell} + e^{-\gamma\ell})}{-2Z_oV_o} = \cosh(\gamma\ell)$$

continue

The complete matrix is therefore

$$\begin{bmatrix} \cosh(\gamma\,\ell) & Z_o \sinh(\gamma\,\ell) \\ \dfrac{\sinh(\gamma\,\ell)}{Z_o} & \cosh(\gamma\,\ell) \end{bmatrix}$$

When the transmission line is lossless this reduces to

$$\begin{bmatrix} \cos(k\,\ell) & jZ_o \sin(k\,\ell) \\ j\dfrac{\sin(k\,\ell)}{Z_o} & \cos(k\,\ell) \end{bmatrix}$$

Note that

$$\boxed{\gamma = \alpha + jk}$$

Where
α= attenuation
k=wave propagation constant

Lossless line
$\alpha = 0$

$$\cosh(jk\,\ell) = \cos(k\ell)$$

$$\sinh(jk\,\ell) = j\sin(k\ell)$$

Table of ABCD network

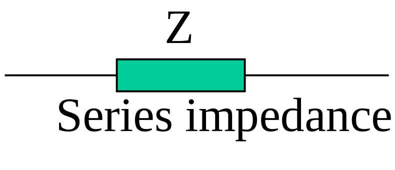
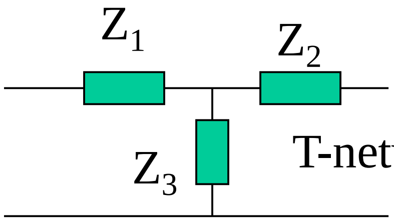
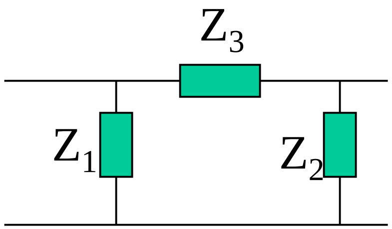
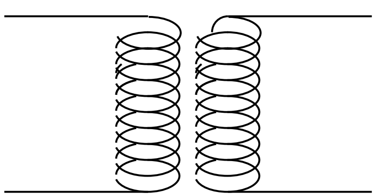
	$\begin{bmatrix} \cosh(\gamma \ell) & Z_o \sinh(\gamma \ell) \\ \dfrac{\sinh(\gamma \ell)}{Z_o} & \cosh(\gamma \ell) \end{bmatrix}$
	$\begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$
	$\begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z} & 1 \end{bmatrix}$

Table of ABCD network

 T-network	$\begin{bmatrix} 1+\dfrac{Z_1}{Z_2} & Z_1+Z_2+\dfrac{Z_1Z_2}{Z_3} \\ \dfrac{1}{Z_3} & 1+\dfrac{Z_2}{Z_3} \end{bmatrix}$
 π-network	$\begin{bmatrix} 1+\dfrac{Z_3}{Z_2} & Z_3 \\ \dfrac{1}{Z_1}+\dfrac{1}{Z_2}+\dfrac{Z_3}{Z_1Z_2} & 1+\dfrac{Z_3}{Z_1} \end{bmatrix}$
 n:1	$\begin{bmatrix} n & 0 \\ 0 & \dfrac{1}{n} \end{bmatrix}$ Ideal transformer

Short transmission line

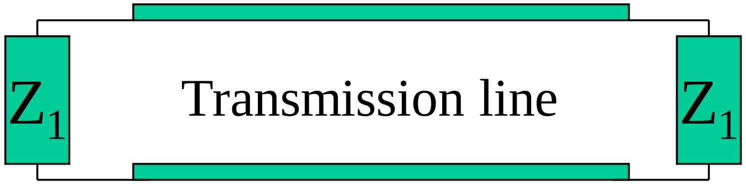
Lossless transmission line

$$ABCD\Big|_{tline} = \begin{bmatrix} \cos(k\ell) & jZ_o \sin(k\ell) \\ j\dfrac{\sin(k\ell)}{Z_o} & \cos(k\ell) \end{bmatrix}$$

If $\ell << \lambda$ then $\cos(k\ell) \simeq 1$ and $\sin(k\ell) \simeq k\ell$ then

$$ABCD\Big|_{tlineshort} = \begin{bmatrix} 1 & jZ_o k\ell \\ j\dfrac{1}{Z_o}k\ell & 1 \end{bmatrix}$$

Embedded short transmission line



$$ABCD\Big|_{embed} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & jZ_o k\ell \\ j\dfrac{1}{Z_o}k\ell & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_1} & 1 \end{bmatrix}$$

Solving, we have

$$ABCD\Big|_{embed} = \begin{bmatrix} 1+\dfrac{jZ_o k\ell}{Z_1} & jZ_o k\ell \\ \dfrac{2}{Z_1}+\dfrac{jZ_o k\ell}{Z_1{}^2}+j\dfrac{k\ell}{Z_o} & 1+\dfrac{jZ_o k\ell}{Z_1} \end{bmatrix}$$

Comparison with π-network

$$ABCD_{\pi-net} = \begin{bmatrix} 1+\dfrac{Z_3}{Z_2} & Z_3 \\ \dfrac{1}{Z_1}+\dfrac{1}{Z_2}+\dfrac{Z_3}{Z_1Z_2} & 1+\dfrac{Z_3}{Z_1} \end{bmatrix}$$

$$ABCD\Big|_{embed} = \begin{bmatrix} 1+\dfrac{jZ_o k\ell}{Z_1} & jZ_o k\ell \\ \dfrac{2}{Z_1}+\dfrac{jZ_o k\ell}{Z_1^{\,2}}+j\dfrac{k\ell}{Z_o} & 1+\dfrac{jZ_o k\ell}{Z_1} \end{bmatrix}$$

It is interesting to note that if we substitute in ABCD matrix in π-network, $Z_2=Z_1$ and $Z_3=jZ_o k\ell$ we see that the difference is in C element where we have extra term i.e $j\dfrac{k\ell}{Z_o}$

Both are almost same if $\dfrac{Z_o k\ell}{Z_1^{\,2}} >> \dfrac{k\ell}{Z_o}$

So the transmission line exhibit a π-network

Comparison with series and shunt

Series

If $Z_o >> Z_1$ then the series impedance $Z = jZ_o k\ell$

This is an inductance which is given by $L = \dfrac{Z_o \ell}{c}$

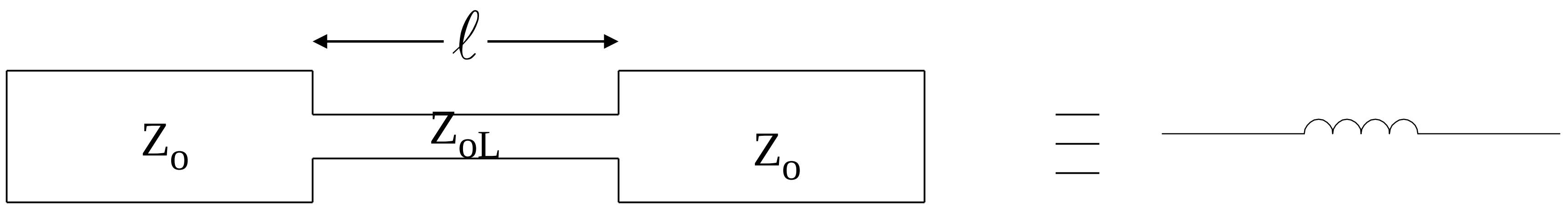
Where c is a velocity of light

Shunt

If $Z_o << Z_1$ then the series impedance $Z = j\dfrac{k\ell}{Z_o}$

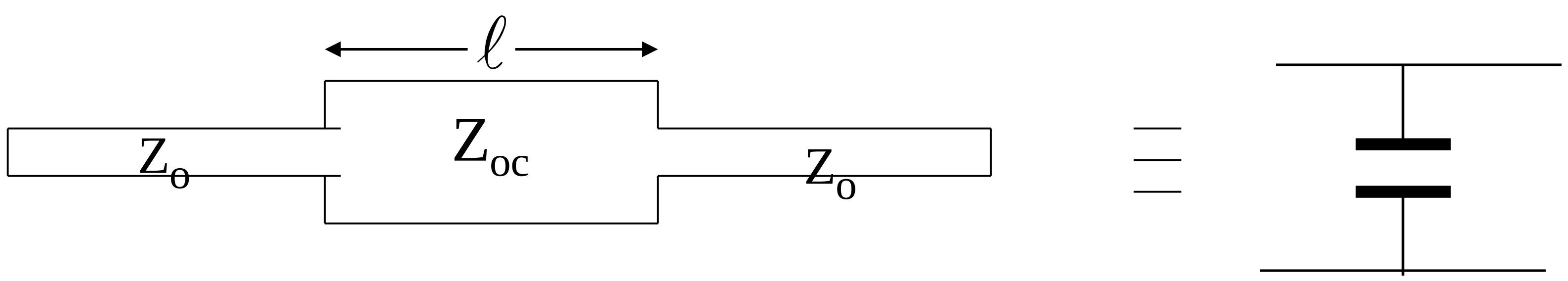
This is a capacitance which is given by $C = \dfrac{\ell}{Z_o c}$

Equivalent circuits



$Z_o >> Z_1$

$$L = \frac{Z_o\,\ell}{c}$$



$Z_o << Z_1$

$$C = \frac{\ell}{Z_o c}$$

Transmission line parameters

It is interesting that the characteristic impedance and propagation constant of a transmission line can be determined from ABCD matrix as follows

$$Z_o = \sqrt{\frac{B}{C}}$$

$$\gamma = \frac{1}{\ell}\cosh^{-1}(A) = \frac{1}{\ell}\ln\left(A \pm \sqrt{A^2 - 1}\right)$$

Conversion S to ABCD

For conversion of ABCD to S-parameter

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \frac{1}{Z_o A + B + Z_o^2 C + Z_o D} \begin{bmatrix} Z_o A + B - Z_o^2 C - Z_o D & 2Z_o (AD - BC) \\ 2Z_o & -Z_o A + B - Z_o^2 C + Z_o D \end{bmatrix}$$

For conversion of S to ABCD-parameter

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{2S_{21}} \begin{bmatrix} (1+S_{11})(1-S_{22}) + S_{12}S_{21} & Z_o\left((1+S_{11})(1+S_{22}) - S_{12}S_{21}\right) \\ \frac{1}{Z_o}\left((1-S_{11})(1-S_{22}) - S_{12}S_{21}\right) & (1-S_{11})(1+S_{22}) + S_{12}S_{21} \end{bmatrix}$$

Z_o is a characteristic impedance of the transmission line connected to the ABCD network, usually 50 ohm.

MathCAD functions for conversion

For conversion of ABCD to S-parameter

$$S(A)=\frac{1}{Z.A_{1,1}+A_{1,2}+Z.Z.A_{2.1}+Z.A_{2,2}}\begin{bmatrix} Z.A_{1,1}+A_{1,2}-Z.Z.A_{2,1}-Z.A_{2,2} & 2.Z.\left(A_{1,1}.A_{2,2}-A_{1,2}.A_{2,1}\right) \\ 2.Z & -Z.A_{1,1}+A_{1,2}-Z.Z.A_{2,1}+Z.A_{2,2} \end{bmatrix}$$

For conversion of S to ABCD-parameter

$$A(S)=\frac{1}{2.S_{2,1}}.\begin{bmatrix} \left(1+S_{1,1}\right).\left(1-S_{2,2}\right)+S_{1,2}.S_{2,1} & Z.\left(\left(1+S_{1,1}\right).\left(1+S_{2,2}\right)-S_{1,2}.S_{2,1}\right) \\ \left(\frac{1}{Z_o}\right).\left(\left(1-S_{1,1}\right).\left(1-S_{2,2}\right)-S_{1,2}.S_{2,1}\right) & \left(1-S_{1,1}\right).\left(1+S_{2,2}\right)+S_{1,2}.S_{2,1} \end{bmatrix}$$

Odd and Even Mode Analysis

Usually use for analyzing a symmetrical four port network

(1) Excitation
- Equal ,in phase excitation – even mode
- Equal ,out of phase excitation – odd mode

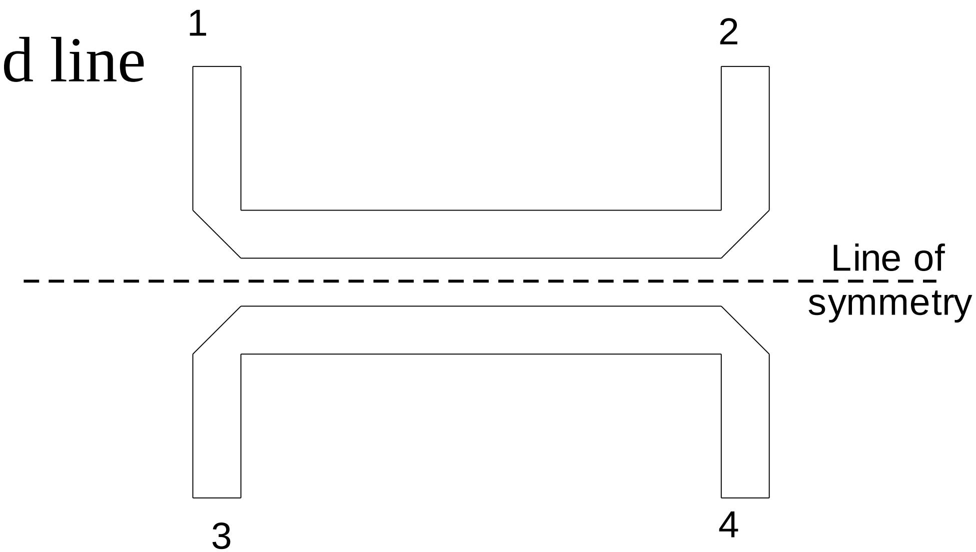
(2) Draw intersection line for symmetry and apply
- short circuit for odd mode
- Open circuit for even mode

(3) Also can apply EM analysis of structure
- Tangential E field zero – odd mode
- Tangential H field zero – even mode

(4) Single excitation at one port= even mode + odd mode

Example 1

Edge coupled line



The matrix contains the odd and even parts

$$[S]=\frac{1}{2}\begin{bmatrix} S_{11ev}+S_{11od} & S_{12ev}+S_{12od} & S_{13ev}-S_{13od} & S_{14ev}-S_{14od} \\ S_{21ev}+S_{21od} & S_{22ev}+S_{22od} & S_{23ev}-S_{23od} & S_{24ev}-S_{24od} \\ S_{31ev}-S_{31od} & S_{32ev}-S_{32od} & S_{33ev}+S_{33od} & S_{34ev}+S_{34od} \\ S_{41ev}-S_{41od} & S_{42ev}-S_{42od} & S_{43ev}+S_{43od} & S_{44ev}+S_{44od} \end{bmatrix}$$

Since the network is symmetry, Instead of 4 ports , we can only analyze 2 port

continue

We just analyze for 2 transmission lines with characteristic Z_e and Z_o respectively. Similarly the propagation coefficients β_e and β_o respectively. Treat the odd and even mode lines as uniform lossless lines. Taking ABCD matrix for a line , length l, characteristic impedance Z and propagation constant β,thus

$$ABCD\Big|_{tline} = \begin{bmatrix} \cos(\beta\ell) & jZ\sin(\beta\ell) \\ j\dfrac{\sin(\beta\ell)}{Z} & \cos(\beta\ell) \end{bmatrix}$$

Using conversion

$$[S] = \frac{1}{Z_o A + B + Z_o^2 C + Z_o D} \begin{bmatrix} Z_o A + B - Z_o^2 C - Z_o D & 2Z_o(AD - BC) \\ 2Z_o & -Z_o A + B - Z_o^2 C + Z_o D \end{bmatrix}$$

continue

$$[S]=\frac{1}{2Z\cos\beta\ell+j\sin\beta\ell\left(\frac{Z^2+Z_o^2}{Z^2}\right)}\begin{bmatrix} j\sin\beta\ell\left(\frac{Z^2-Z_o^2}{Z}\right) & 2Z_o \\ 2Z_o & j\sin\beta\ell\left(\frac{Z^2-Z_o^2}{Z}\right)\end{bmatrix}$$

Taking $\beta\ell=\frac{\pi}{2}$ (equivalent to quarter-wavelength transmission line)

Then

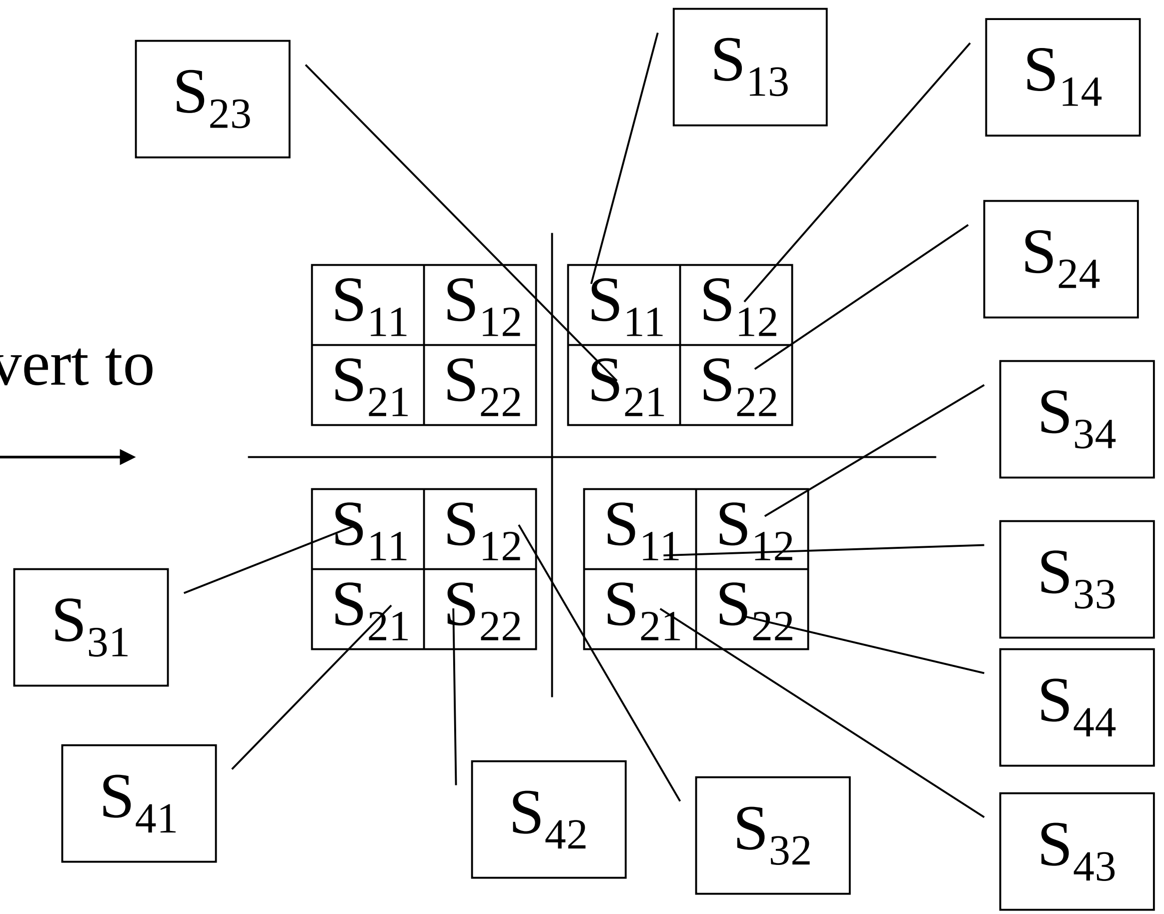
$$[S]=\frac{1}{Z^2+Z_o^2}\begin{bmatrix} Z^2-Z_o^2 & -j2ZZ_o \\ -j2ZZ_o & Z^2-Z_o^2\end{bmatrix}$$

continue

Odd ± even

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Convert to $\longrightarrow$



2-port network matrix

4-port network matrix

continue

Follow symmetrical properties

$$\begin{array}{c|c|c|c}
\text{ev+ od} & \begin{array}{|c|c|}\hline S_{11} & S_{12} \\ \hline S_{21} & S_{22} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{13} & S_{14} \\ \hline S_{23} & S_{24} \\ \hline \end{array} & \text{ev- od} \\
\hline
\text{ev- od} & \begin{array}{|c|c|}\hline S_{31} & S_{32} \\ \hline S_{41} & S_{42} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{33} & S_{34} \\ \hline S_{43} & S_{44} \\ \hline \end{array} & \text{ev+ od}
\end{array}$$

Assuming $\beta_{ev} = \beta_{od} = \dfrac{\pi}{2\ell}$

Then

$$S_{41} = S_{14} = S_{32} = S_{23} = -\frac{jZ_o}{2}\left(\frac{Z_{ev}}{Z_{ev}^2 + Z_o^2} - \frac{Z_{od}}{Z_{od}^2 + Z_o^2}\right)$$

$$= -\frac{jZ_o}{2}\left(\frac{(Z_{ev}Z_{od} - Z_o^2)}{(Z_{ev}^2 + Z_o^2)}\frac{(Z_{od} - Z_{ev})}{(Z_{od}^2 + Z_o^2)}\right)$$

For perfect isolation (I.e S_{41}=S_{14}=S_{32}=S_{23}=0),we choose Z_{ev} and Z_{od} such that $Z_{ev}\,Z_{od}$=Z_o^2.

continue

$$\begin{array}{cc|cc}
\text{ev+ od} & \begin{array}{|c|c|}\hline S_{11} & S_{12} \\ \hline S_{21} & S_{22} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{13} & S_{14} \\ \hline S_{23} & S_{24} \\ \hline \end{array} & \text{ev- od} \\
\hline
\text{ev- od} & \begin{array}{|c|c|}\hline S_{31} & S_{32} \\ \hline S_{41} & S_{42} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{33} & S_{34} \\ \hline S_{43} & S_{44} \\ \hline \end{array} & \text{ev+ od}
\end{array}$$

Similarly we have

$$S_{11} = S_{22} = S_{33} = S_{44} = \frac{1}{2}\left(\frac{Z_{ev}^2 - Z_o^2}{Z_{ev}^2 + Z_o^2} + \frac{Z_{od}^2 - Z_o^2}{Z_{od}^2 + Z_o^2} \right)$$

$$= \frac{1}{2}\left(\frac{Z_{ev}^2 Z_{od}^2 - Z_o^4}{(Z_{ev}^2 + Z_o^2)(Z_{od}^2 + Z_o^2)} \right)$$

Equal to zero if $Z_{ev}\, Z_{od} = Z_o^2$.

continue

ev+ od	S_{11}	S_{12}	S_{13}	S_{14}	ev- od
	S_{21}	S_{22}	S_{23}	S_{24}	
ev- od	S_{31}	S_{32}	S_{33}	S_{34}	
	S_{41}	S_{42}	S_{43}	S_{44}	ev+ od

We have

$$S_{31} = S_{13} = S_{24} = S_{42} = \frac{1}{2}\left(\frac{Z_{ev}^2 - Z_o^2}{Z_{ev}^2 + Z_o^2} - \frac{Z_{od}^2 - Z_o^2}{Z_{od}^2 + Z_o^2} \right)$$

$$= \left(\frac{(Z_{ev}^2 - Z_{od}^2) Z_o^2}{(Z_{ev}^2 + Z_o^2)(Z_{od}^2 + Z_o^2)} \right)$$

$$= \left(\frac{Z_{ev} - Z_{od}}{Z_{ev} + Z_{od}} \right)$$

if $Z_{ev}\, Z_{od} = Z_o^2$.

continue

$$\begin{array}{c|c}
\text{ev+ od} \quad \begin{array}{|c|c|}\hline S_{11} & S_{12} \\ \hline S_{21} & S_{22} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{13} & S_{14} \\ \hline S_{23} & S_{24} \\ \hline \end{array} \quad \text{ev- od} \\
\hline
\text{ev- od} \quad \begin{array}{|c|c|}\hline S_{31} & S_{32} \\ \hline S_{41} & S_{42} \\ \hline \end{array} & \begin{array}{|c|c|}\hline S_{33} & S_{34} \\ \hline S_{43} & S_{44} \\ \hline \end{array} \quad \text{ev+ od}
\end{array}$$

$$S_{21} = S_{12} = S_{34} = S_{43} = -\frac{jZ_o}{2}\left(\frac{Z_{ev}}{Z_{ev}^2 + Z_o^2} + \frac{Z_{od}}{Z_{od}^2 + Z_o^2}\right)$$

$$= -jZ_o\left(\frac{1}{Z_{ev} + Z_{od}}\right)$$

if $Z_{ev}\, Z_{od} = Z_o^2$.

continue

This S-parameter must satisfy network characteristic:

(1) Power conservation

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 1$$

Reflected power

transmitted power to port 2

transmitted power to port 3

transmitted power to port 4

Since S_{11} and S_{41}=0 , then

$$|S_{21}|^2 + |S_{31}|^2 = 1$$

(2) And quadrature condition $$Arg\left(\frac{S_{11}}{S_{21}}\right) = \pm\frac{\pi}{2}$$

continue

For 3 dB coupler

$$\left(\frac{Z_{ev}-Z_{od}}{Z_{ev}+Z_{od}}\right)^2=\frac{1}{2} \qquad \text{or} \qquad \left(\frac{Z_{ev}-Z_{od}}{Z_{ev}+Z_{od}}\right)=\pm\sqrt{\frac{1}{2}}$$

Rewrite we have

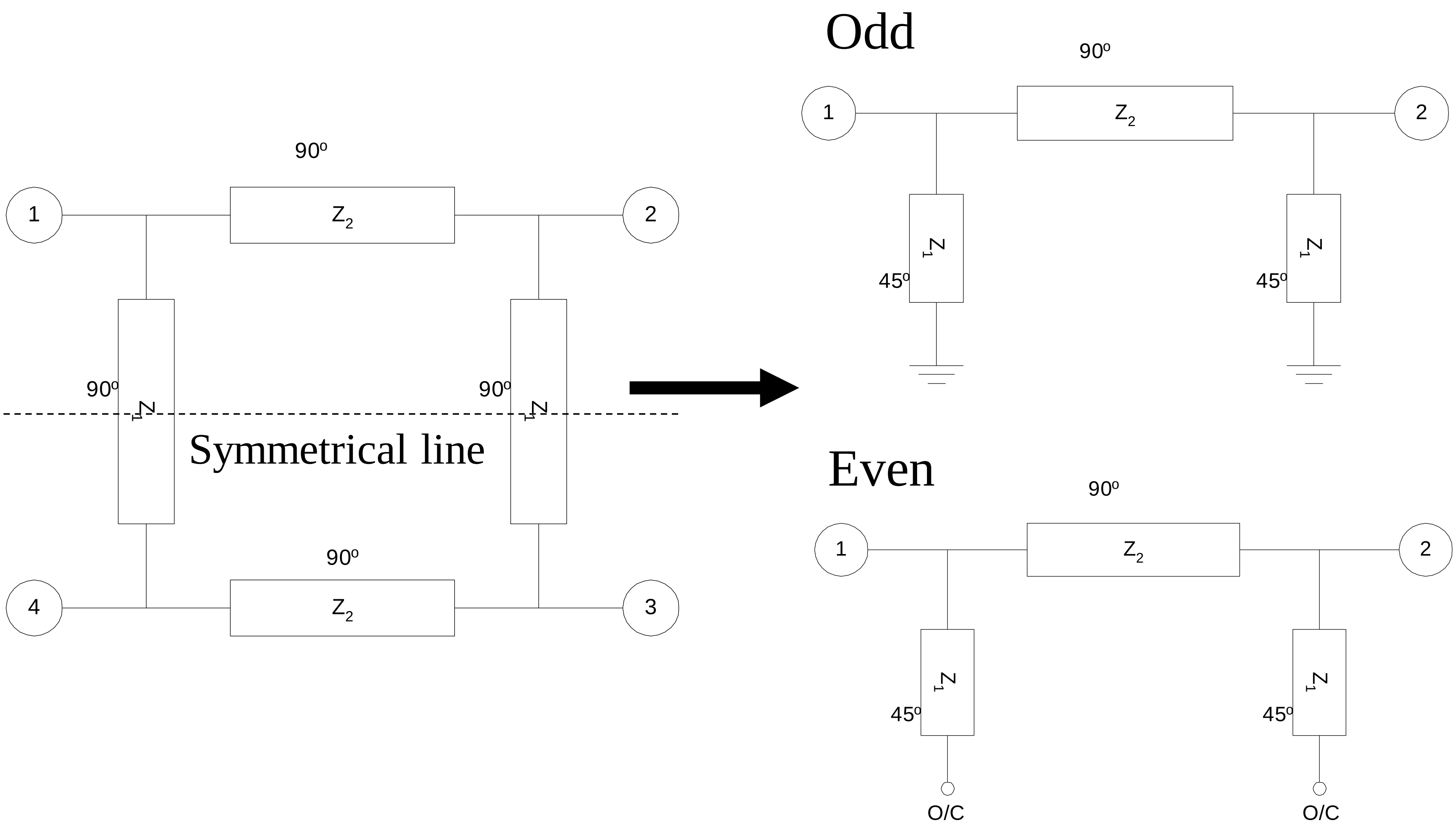
$$\frac{Z_{ev}}{Z_{od}}=\left(\frac{1+(\pm\sqrt{2})}{1-(\pm\sqrt{2})}\right)=3\pm2\sqrt{2}$$

In practice $Z_{ev} > Z_{od}$ so $\frac{Z_{ev}}{Z_{od}}=3+2\sqrt{2}=5.83$

However the limitation for coupled edge

$\frac{Z_{ev}}{Z_{od}}<2$ (Gap size) also β_{ev} and β_{od} are not pure TEM thus not equal

A λ/4 branch line coupler



Analysis

Stub odd (short circuit) $$X_{s,od} = Z_1 \tan\left(\frac{\pi}{4}\right) = Z_1$$

Stub even (open circuit) $$X_{s,ev} = -Z_1 \cot\left(\frac{\pi}{4}\right) = Z_1$$

The ABCD matrices for the two networks may then found :

$$ABCD\Big| = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{jX_s} & 1 \end{bmatrix} \begin{bmatrix} 0 & jZ_2 \\ \dfrac{j}{Z_2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{jX_s} & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{Z_2}{X_s} & jZ_2 \\ \dfrac{j}{Z_2} - \dfrac{jZ_2}{X_s^2} & \dfrac{Z_2}{X_s} \end{bmatrix}$$

stub

Transmission line

stub

continue

Convert to S

$$[S]=\frac{1}{Z_oA+B+Z_o^2C+Z_oD}\begin{bmatrix} Z_oA+B-Z_o^2C-Z_oD & 2Z_o(AD-BC) \\ 2Z_o & -Z_oA+B-Z_o^2C+Z_oD \end{bmatrix}$$

$$=\frac{1}{\frac{2Z_oZ_2}{X_s}+jZ_2-j\frac{Z_o^2Z_2}{X_s^2}+j\frac{Z_o^2}{Z_2}}\begin{bmatrix} jZ_2+j\frac{Z_o^2Z_2}{X_s^2}-j\frac{Z_o^2}{Z_2} & 2Z_o \\ 2Z_o & jZ_2+j\frac{Z_o^2Z_2}{X_s^2}-j\frac{Z_o^2}{Z_2} \end{bmatrix}$$

For perfect isolation we require

$$S_{11ev}+S_{11od}=S_{11ev}-S_{11od}=0$$ Thus $$S_{11ev}=S_{11od}=0$$

$$S_{11}=jZ_2+j\frac{Z_o^2Z_2}{X_s^2}-j\frac{Z_o^2}{Z_2}=0$$ or $$X_s=\mp\frac{Z_oZ_2}{\sqrt{Z_o^2-Z_2^2}}=Z_1$$

From previous definition

continue

Substituting into S-parameter gives us

$$[S]_{odd} = \frac{1}{\sqrt{Z_o^2 - Z_2^2 + jZ_2}} \begin{bmatrix} 0 & Z_o \\ Z_o & 0 \end{bmatrix} \quad \text{and} \quad [S]_{even} = -\frac{1}{\sqrt{Z_o^2 - Z_2^2 + jZ_2}} \begin{bmatrix} 0 & Z_o \\ Z_o & 0 \end{bmatrix}$$

Therefore for full four port

$$S_{21} = S_{12} = S_{43} = S_{34} = \frac{1}{2}\left(S_{21ev} + S_{21od}\right) = -j\,\frac{Z_2}{Z_o}$$

$$S_{41} = S_{14} = S_{32} = S_{23} = \frac{1}{2}\left(S_{21ev} - S_{21od}\right) = -\sqrt{1 - \frac{{Z_2}^2}{{Z_o}^2}}$$

$$S_{11} = S_{22} = S_{33} = S_{44} = 0$$

And $\quad S_{31} = S_{13} = S_{42} = S_{24} = 0$

continue

For power conservation and quadrature conditions to be met

Equal split S

$$|S_{21}| = \frac{Z_2}{Z_o} = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad Z_2 = \frac{Z_o}{\sqrt{2}}$$

And

$$X_s = Z_1 = \frac{Z_o Z_2}{\sqrt{Z_o^2 - Z_2^2}} = \frac{Z_o \dfrac{Z_o}{\sqrt{2}}}{\sqrt{Z_o^2 - \left(\dfrac{Z_o}{\sqrt{2}}\right)^2}} = Z_o$$

If $Z_o = 50\ \Omega$ then $Z_2 = 35.4\ \Omega$