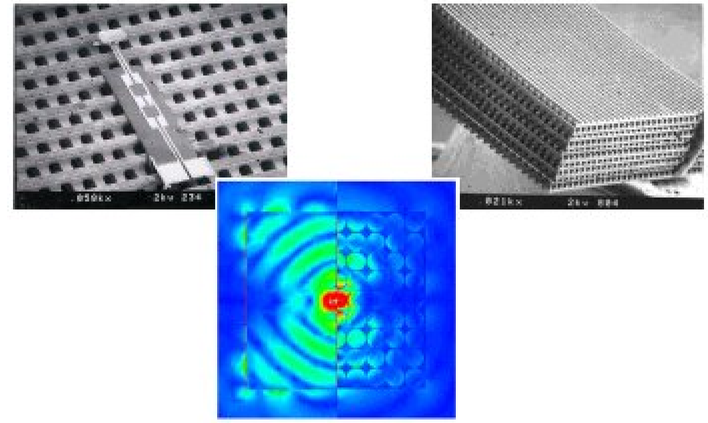
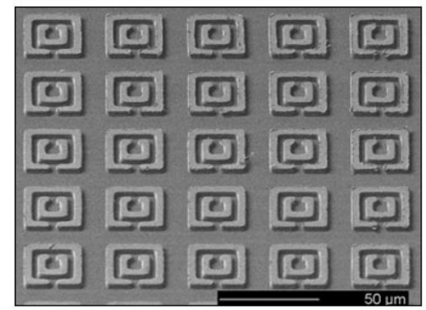
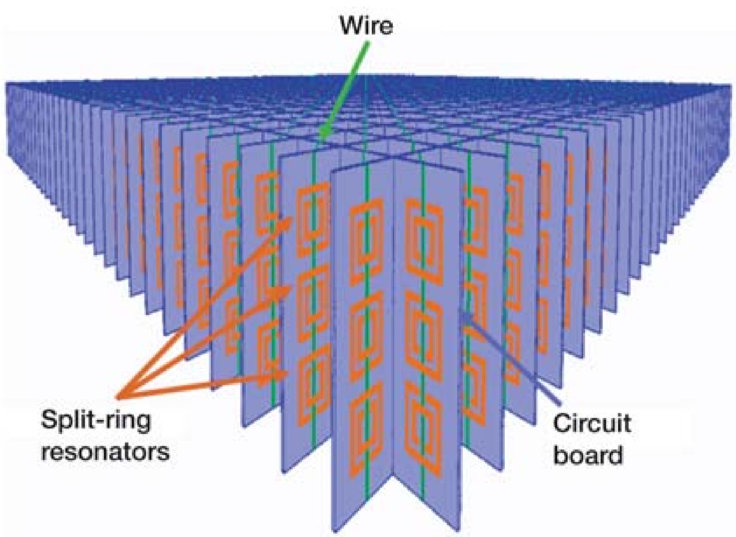
Periodic Structures and Floquet's Theorem



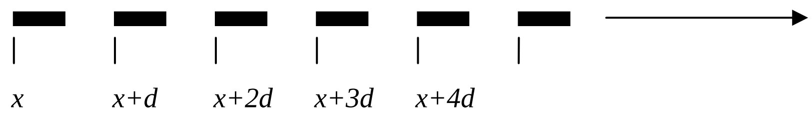
- Periodic Structures
  - o Repeated geometry, defined by a "unit cell" and a uniform periodic spacing
- Applications
  - o Periodic Array antennas
  - o High-impedance surfaces
  - o Frequency Selective Surfaces (FSS)
  - o Meta-Materials
    - Artificial Materials
    - Left-handed materials
    - Chiral materials
    - Frequency selective material behavior





1D Periodic Surface

- Consider a 1D periodic surface (in *x*)
  - o Structure is infinite in $\pm x$ and periodic
  - o Period d



- Let $u(x)$ represent a field reacting with the periodic surface
  - o Geometric periodicity forces the field to be periodic

$$
\begin{aligned}
u(x+d) &= Cu(x) \\
u(x+2d) &= Cu(x+d) \\
u(x+3d) &= Cu(x+2d) \\
&\vdots \\
u(x+nd) &= Cu(x+(n-1)d)
\end{aligned}
$$

    - ▪ C = a complex constant
  - o More generally:

$$u(x+nd) = C^n u(x)$$

  - o For boundedness, $|C| \le 1$
  - o In general: $C = e^{+jkd}$, k = complex constant

Periodic Function

- We can define a periodic function $P(x)$, where

$$P(x) = e^{-jkx}u(x)$$

- Consequently

$$P(x+d) = e^{-jk(x+d)}u(x+d) = e^{-jk(x+d)}Cu(x) = e^{-jkx}e^{-jkd}\left(e^{jkd}\right)u(x)$$

$$= e^{-jkx}u(x) = P(x)$$

- Similarly:

$$P(x+nd) = P(x)$$

- $P(x)$ is a periodic function in *x*, with period *d*
  - o $P(x)$ has the same period as the geometry
  - o Note that the periodic phase shift and attenuation is normalized out in $P(x)$.
  - o Since $P(x)$ is periodic in *x*, we can express it via a Fourier Series expansion:

$$P(x) = \sum_{n=-\infty}^{\infty} p_n e^{j\frac{2\pi n}{d}x}$$

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- Substituting: $P(x) = e^{-jkx}u(x)$, then

$$u(x) = \sum_{n=-\infty}^{\infty} p_n e^{jkx} e^{j\frac{2\pi n}{d}x} \Rightarrow$$

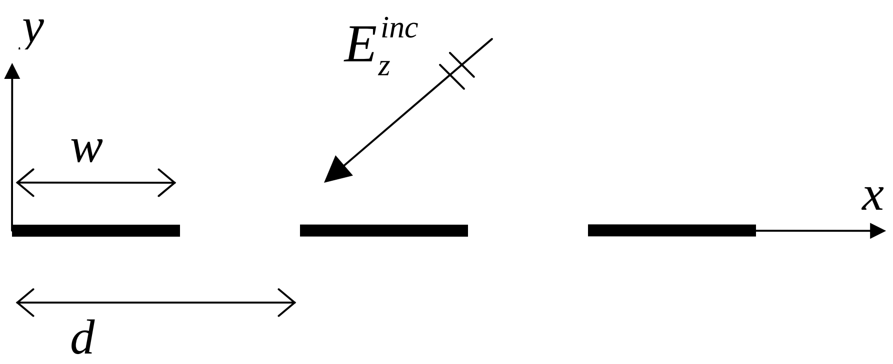
$$u(x) = \sum_{n=-\infty}^{\infty} p_n e^{jk_{x_n}x}$$

- where

$$k_{x_n} = k + \frac{2\pi n}{d}$$

- This represents a harmonic expansion of the field *u(x)*.
- Each term represents a spatial *Floquet Harmonic*
  - Infinite series
  - All terms propagate along the periodic axis (forward *and* backward waves)
- Based on Floquet's theorem, any planar periodic function can be expanded as an infinite superposition of Floquet harmonics. Here it is presented for a 1D periodic structure. It can be easily generalized to higher dimensions with more complex periodicities.

1D Periodic Strip Grating



- Consider a TMz polarized plane wave incident on a co-planar periodic grating of metallic strips. The strips have a width *w*. The grating has a period *d*.
- Incident Electric field:

$$E_z^{inc}(x,y) = e^{j\vec{k}\cdot\vec{r}} = e^{jk_0\left(x\cos\phi^{inc}+y\sin\phi^{inc}\right)}$$

  - ϕ^{inc} = the angle of the wave off the *x*-axis.
- Following Floquet's theorem, we can express the incident plane wave as a periodic function in *d*

$$E_z^{inc}(x+d,y) = e^{jk_0\left((x+d)\cos\phi^{inc}+y\sin\phi^{inc}\right)} = e^{jk_o d\cos\phi^{inc}} e^{jk_0\left(x\cos\phi^{inc}+y\sin\phi^{inc}\right)}$$

$$= E_z^{inc}(x,y)\, e^{jk_o d\cos\phi^{inc}}$$

- Therefore, the complex constant $C = e^{jk_o d\cos\phi^{inc}}$.
  - and

$$k = k_o \cos\phi^{inc}$$

- Our objective is to compute the field scattered by the periodic strip grating.
  - Pose the scattered field as a function of Floquet Harmonics:

$$E_z^{scat}(x,y) = \sum_{n=-\infty}^{\infty} a_n(y) e^{jk_{x_n}x}$$

  - Each harmonic must satisfy the wave equation
  - The sum of harmonics is subject to the boundary conditions on the metallic strip surfaces $\left. E_z^{tot}(x,0)\right|_{nd\le x\le nd+w} = 0$
    - Off the PEC surface, E_z and H_x are continuous
  - The coefficient of each harmonic is a function if *y* due to the boundary condition at $y = 0$.
  - The coefficients are independent of *z* since the source and the geometry are independent of *z*.
- The scattered field must satisfy the wave equation:

$$\nabla^2 E_z^{scat} + k_0^2 E_z^{scat} = 0, \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_0^2\right) E_z^{scat} = 0$$

- Therefore, we expect $a_n(y)$ to be of the form:

$$a_n(y) = c_n e^{\mp jk_{y_n} y}, \qquad \{(-)\ y > 0,\ (+)\ y < 0\}$$

- Plugging this into the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_0^2\right) c_n e^{\mp jk_{y_n} y} e^{jk_{x_n} x} = 0$$

$$\left(k_0^2 - k_{x_n}^2 - k_{y_n}^2\right) = 0$$

$$k_{y_n} = \sqrt{k_0^2 - k_{x_n}^2}$$

- This is referred to as the *dispersion relationship*.
- Therefore,

$$E_z^{scat}(x, y) = \sum_{n=-\infty}^{\infty} c_n e^{jk_{x_n} x} e^{-jk_{y_n}|y|}$$

- Thus, the scattered field is expanded via a *plane wave expansion* of upward and downward traveling waves.
- The coefficients c_n are yet to be determined.
  - o These represent the complex amplitudes of each harmonic, are will be based on the boundary conditions on the strip surface
  - o We will determine these coefficients via a method of moment procedure.

- Note that the angular spectrum of the Floquet Harmonic expansion is independent of the coefficients c_n.
- Recall that:

$$k_{x_n} = k + \frac{2\pi n}{d} = k_o \cos\phi^{inc} + \frac{2\pi n}{d}$$

- Therefore,

$$k_{y_n} = \sqrt{k_0^2 - \left(k_o \cos\phi^{inc} + \frac{2\pi n}{d} \right)^2}$$

- Since n varies from $-\infty$ to $+\infty$, k_{x_n} varies dramatically between $-\infty$ and $+\infty$.
- Define:

$$k_{y_n} = \begin{cases} \sqrt{k_0^2 - k_{x_n}^2}, & k_0^2 \geq k_{x_n}^2 \\ -j\sqrt{k_{x_n}^2 - k_0^2}, & k_0^2 \leq k_{x_n}^2 \end{cases}$$

- Assuming k_0 is real, then k_{y_n} is defined by a finite set of purely propagating modes, and an infinite set of evanescent modes.

- Note that *multiple* propagating modes can be scattered off the periodic surface. The angle of the propagating modes are defined as:

$$k_{x_n} = k_o\left(\cos\phi^{inc} + \frac{n}{d/\lambda_0}\right), \quad k_{y_n} = k_0\sqrt{1-\left(\cos\phi^{inc} + \frac{n}{d/\lambda_0}\right)^2}$$

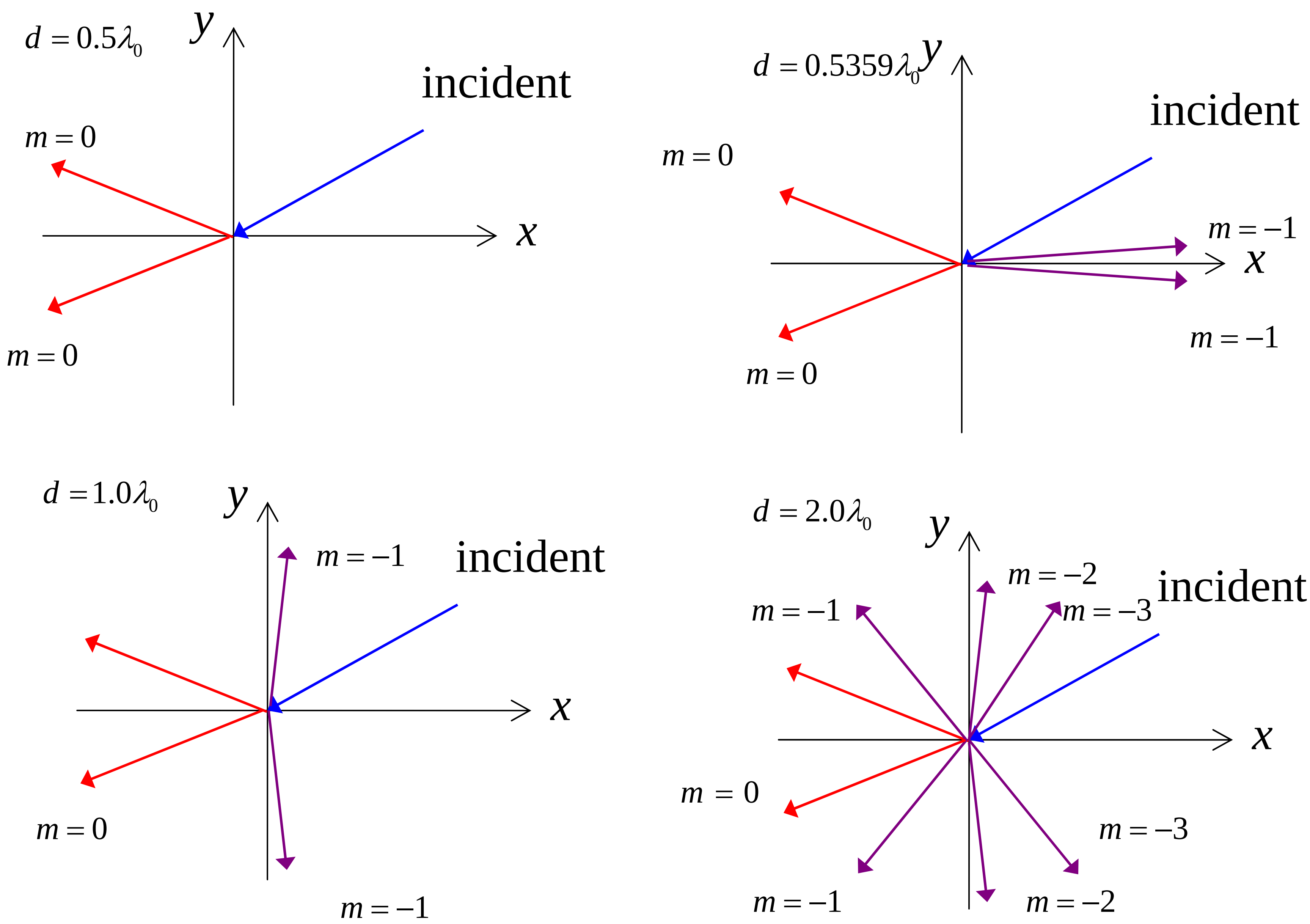
$$\phi^s = \tan^{-1}\left(\frac{k_{y_n}}{k_{x_n}}\right)$$

- Examples:
  - $\phi^{inc} = 30°$

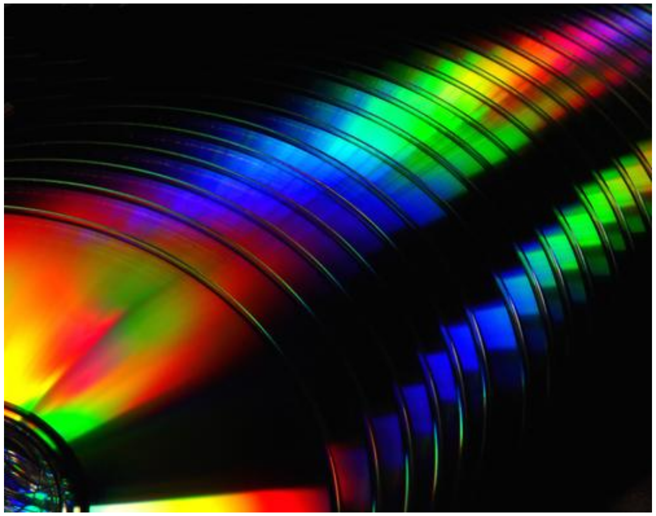
d	n	k_{x_n}/k_0	k_{y_n}/k_0	$\phi_n^{scat} = \tan^{-1}\left(k_{y_n}/k_{x_n}\right)$[1]
$0.5\lambda_0$	-1	-1.134	$0.535j$	Evanescent
	0	0.866	0.5	30^o
	1	2.866	$2.686j$	Evanescent
$0.5359\lambda_0$	-1	-0.192	0.003354	179.8^o
	0	0.866	0.5	30^o
	1	2.732	$2.542j$	Evanescent

[1] The scattered angle is off the *negative* x-axis

d	n	k_{x_n} / k_0	k_{y_n} / k_0	$\phi_n^{scat} = \tan^{-1}\left(k_{y_n} / k_{x_n}\right)$
$1.0\lambda_0$	-2	-1.134	$0.535j$	Evanescent
	-1	-0.134	0.991	97.7^o
	0	0.866	0.5	30^o
	1	1.866	$1.575j$	Evanescent
$2.0\lambda_0$	-4	-1.134	$0.535j$	Evanescent
	-3	-0.634	0.773	-129.3^o
	-2	-0.134	0.991	-97.7^o
	-1	0.366	0.931	68.6^o
	0	0.866	0.5	30^o
	1	1.366	$-0.931j$	Evanescent


$d = 0.5\lambda_0$
y
incident
$m = 0$
x
$m = 0$
$d = 0.5359\lambda_0$
y
incident
$m = 0$
$m = -1$
x
$m = -1$
$m = 0$
$d = 1.0\lambda_0$
y
$m = -1$
incident
x
$m = 0$
$m = -1$
$d = 2.0\lambda_0$
y
$m = -2$
incident
$m = -1$
$m = -3$
x
$m = 0$
$m = -3$
$m = -1$
$m = -2$

- Observations
  - For very small cells ($< \lambda_0 / 2$), only the specular reflection ($n = 0$) will be non-evanescent.
  - As the unit cell size is increased, new propagating modes will appear. These modes will begin propagating along the +ve x-axis, and as frequency increase, will migrate towards the specular reflection.
    - Note that ϕ^{inc} also impacts the frequency at which higher-order harmonics will begin to propagate, and the reflection angle.
  - The harmonics propagate off at discrete angles
    - Practical application:
      - Diffraction Grating
      - Frequency selective surface
      - Dichroic reflector antenna
  - Evanescent modes are *surface wave modes*
    - Propagate along the surface of the structure
    - Purely attenuate away from the surface



1D Periodic Green Function

- The total electric field must satisfy the boundary condition that E_z^{tot} *on the strip surface* = 0.
  - Can also be expressed that $E_z^{scat} = -E_z^{inc}$ on the strip surface
- E_z^{scat} is represented by an infinite summation of Floquet harmonics.
  - We need to solve for the amplitudes of each harmonic to satisfy the boundary condition.
  - This can be done in the same way as performing a Fourier series expansion of a function.
  - The difficulty is the function is a rectangular pulse, which needs an infinite number of harmonics to converge.
- An alternate approach is to derive a Green function for a 1D periodic line source, and then pursue a method of moment solution using the EFIE.
  - This is referred to as the periodic Green function (PGF).

Periodic Green Function

- Consider an infinite 1D array of electric line sources that are illuminated by a plane wave
  - The line sources are effectively radiating a scattered field due the plane wave illuminating the periodic structure.
  - Floquet's theorem states that all observables will have the same periodicity as the structure, and have a cell-to-cell phase shift equal to that of the source.
  - The phase shift will be $k_0 d \cos\phi^{inc}$
- This leads to an array of infinite line sources, periodically placed with spacing d, and with phase shift: $e^{jn\psi} = e^{jnk_0 d\cos\phi^{inc}}$
- The periodic Green function (PGF) satisfies the wave equation:

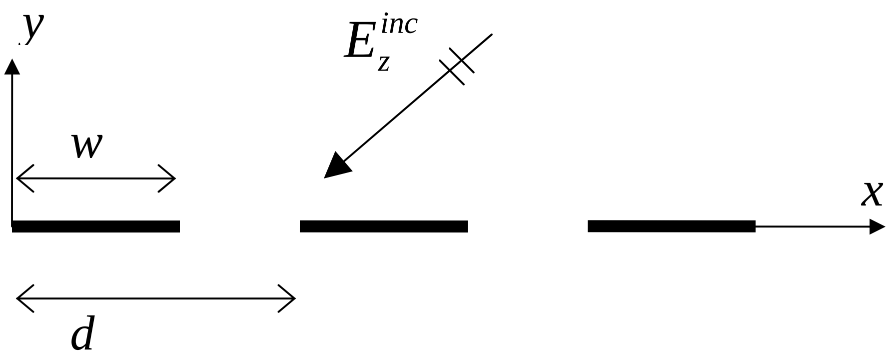
$$\nabla^2 G_p + k_0^2 G_p = -\sum_{n=-\infty}^{\infty} \delta\left(x - nd\right) e^{jn\psi}$$

- We can apply superposition, and solve for one mode at a time:

$$G_p\left(\vec{r}\right) = \frac{1}{4j} \sum_{n=-\infty}^{\infty} H_0^{(2)}\left(k_0 \sqrt{\left(x - nd\right)^2 + y^2}\right) e^{jn\psi}$$

EFIE Solution for the Scattering by a 1D Periodic Strip Array

- Now, consider the scattering by a 1D periodic array of PEC strips co-planar in the y = 0 plane.



- The EFIE is formulated as:

$$E_z^{scat}\left(x,0\right)\Big|_{nd\le x\le nd+w} = -E_z^{inc}\left(x,0\right)\Big|_{nd\le x\le nd+w}$$

- where,

$$E_z^{scat} = -jk_0\eta_0 A_z$$

$$A_z = \int_0^w J_z\left(x'\right) G_p\left(x,y\,|\,x'\right) dx'$$

$$G_p\left(x,y\,|\,x'\right) = \frac{1}{4j}\sum_{n=-\infty}^{\infty} H_0^{(2)}\left(k_0\sqrt{\left(x-x'-nd\right)^2+y^2}\right)e^{jn\psi}$$

- Note that the integration is only over 1 cell.
- This cell is referred to as the *unit cell*.
- The current from the unit cell is repeated by the PGF. Namely, the PGF effectively radiates an infinite number of currents from *all* the periodic cells.

Method of Moment Discretization

- The unit cell strip alone is discretized into *N* linear segments.
- The current density can be expanded using pulse basis functions over each linear segment:

$$J_z \approx \sum_{i=1}^{N} \alpha_i P_i\left(x; x_i, x_{i+1}\right)$$

  - Only the current over the *unit cell* must be expanded
  - The PGF effectively replicates each pulse basis function as an infinite number that are periodically displaced
  - The PGF also applies the correct phase shift to every periodically displaced basis function.
- It is sufficient to use delta test functions
  - The test functions are located at the cell centers

$$T_j = \delta\left(x - x_j^c\right)$$

- The EFIE becomes:

$$-E_z^{inc}\left(x_j^c\right) = -\frac{k_0\eta_0}{4}\sum_{i=1}^{N}\alpha_i \int_{x_i}^{x_{i+1}} \sum_{n=-\infty}^{\infty} H_0^{(2)}\left(k_0\sqrt{\left(x_j^c - x' - nd\right)^2 + 0^2}\right)e^{jn\psi}\,dx'$$

- or,

$$E_z^{inc}\left(x_j^c\right) = \frac{k_0\eta_0}{4}\sum_{i=1}^{N}\alpha_i \sum_{n=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} H_0^{(2)}\left(k_0\left|x_j^c - x' - nd\right|\right)e^{jn\psi}\,dx'$$

- This leads to a linear system of equations:

$$\overline{e}^{inc} = \overline{\overline{Z}}\,\overline{\alpha}$$

- where,

$$e_j^{inc} = E_z^{inc}\left(x_j^c\right)$$

$$Z_{j,i} = \frac{k_0\eta_0}{4}\sum_{n=-\infty}^{\infty} e^{jn\psi}\int_{x_i}^{x_{i+1}} H_0^{(2)}\left(k_0\left|x_j^c - x' - nd\right|\right)dx'$$

$$\psi = k_0 d\cos\phi^{inc}$$

$$Z_{j,i} = \frac{k_0 \eta_0}{4} \sum_{n=-\infty}^{\infty} e^{jn\psi} \int_{x_i}^{x_{i+1}} H_0^{(2)}\left(k_0 \left| x_j^c - x' - nd \right|\right) dx'$$

- Observation:
  - Every impedance element requires numerical integration and an infinite summation
  - The convergence of this series is *extremely slow:*

$$\lim_{nd \to \infty} H_0^2\left(k_0 \left|x - x' - nd\right|\right) \approx \sqrt{\frac{j2}{\pi k_0 |n| d}} e^{-jk_0|n|d} = O\left(\frac{1}{\sqrt{|n|}}\right)$$

  - We need to accelerate the rate of convergence.
  - There are a number of methods to do this. We will look at one method based on using the Spectral Periodic Green function

Spectral Periodic Green Function

- Introduce the Fourier transform pair:

$$F\left(f(x)\right) = \tilde{f}(k_x) = \int_{-\infty}^{\infty} f(x) e^{-jk_x x} dx$$

$$F^{-1}\left(\tilde{f}(k_x)\right) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k_x) e^{jk_x x} dk_x$$

- Next, consider again the infinite line source array:

$$J_z(x) = \sum_{n=-\infty}^{\infty} \delta(x - nd) e^{jn\psi}$$

- We can perform the Fourier transform of $J_z(x)$:

$$\tilde{J}_z(k_x) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - nd) e^{jn\psi} e^{-jk_x x} dx$$

$$= \sum_{n=-\infty}^{\infty} e^{jn\psi} e^{-jk_x nd}$$

- We can prove that this is is equivalent to the Fourier *series* expansion:

$$\frac{2\pi}{d} \sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{2\pi n}{d} - \frac{\psi}{d}\right) = \sum_{n=-\infty}^{\infty} e^{jn\psi} e^{-jk_x nd}$$

- Proof

$$\frac{2\pi}{d}\sum_{n=-\infty}^{\infty}\delta\left(k_x-\frac{2\pi n}{d}-\frac{\psi}{d}\right)=\sum_{n=-\infty}^{\infty}a_n e^{-jk_x nd}$$

$$\int_0^{\frac{2\pi}{d}}\frac{2\pi}{d}\sum_{n=-\infty}^{\infty}\delta\left(k_x-\frac{2\pi n}{d}-\frac{\psi}{d}\right)e^{jk_x md}=\int_0^{\frac{2\pi}{d}}\sum_{n=-\infty}^{\infty}a_n e^{-jk_x nd}e^{jk_x md}dx$$

$$\cancel{\frac{2\pi}{d}}e^{j\left(\frac{2\pi 0}{d}+\frac{\psi}{d}\right)md}=\cancel{\frac{2\pi}{d}}a_m$$

$$a_m=e^{jm\psi}$$

$$\therefore\frac{2\pi}{d}\sum_{n=-\infty}^{\infty}\delta\left(k_x-\frac{2\pi n}{d}-\frac{\psi}{d}\right)=\sum_{n=-\infty}^{\infty}e^{jn\psi}e^{-jk_x nd}\quad\text{Q.E.D.}$$

- Therefore,

$$\tilde{J}_z\left(k_x\right)=\frac{2\pi}{d}\sum_{n=-\infty}^{\infty}\delta\left(k_x-\frac{2\pi n}{d}-\frac{\psi}{d}\right)$$

- Now, we can return to the spatial domain via the inverse Fourier transform:

$$J_z(x) = F^{-1}\left(\tilde{J}_z(k_x)\right) = \frac{1}{2\pi}\int_{k_x=-\infty}^{\infty}\left[\frac{2\pi}{d}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{2\pi n}{d} - \frac{\psi}{d}\right)\right]e^{jk_x x}$$

$$= \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{j\left(\frac{2\pi n}{d}+\frac{\psi}{d}\right)x}$$

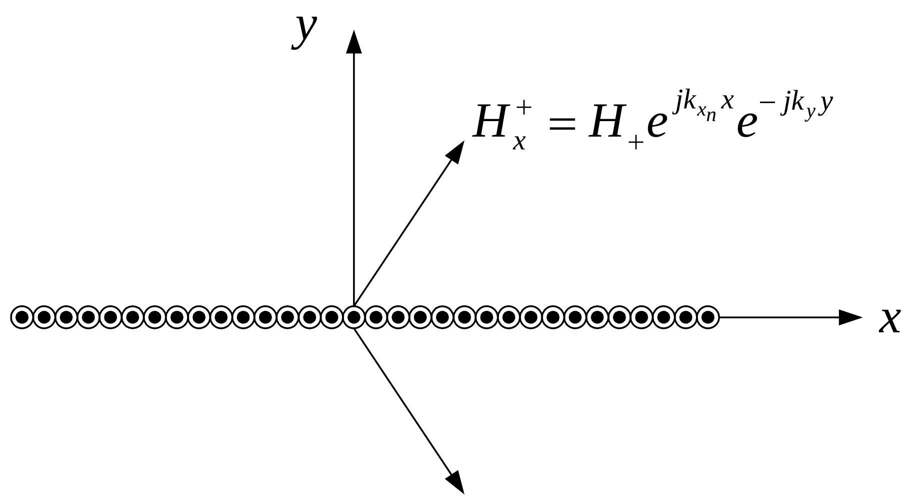
$$= \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{jk_{x_n}x}$$

- Finally:

$$J_z(x) = \sum_{n=-\infty}^{\infty}\delta(x-nd)e^{jn\psi} = \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{jk_{x_n}x}$$

- Observations:
  - The infinite summation of periodically displaced line sources is equal to an infinite summation of current sheets.
  - Note that:
    - Each current sheet physically radiates from the $y = 0$ plane
    - The effective wave number of the current sheets are the Floquet wave numbers
    - Each current sheet is linearly independent
    - The field radiated by each current sheet is a plane wave!

- Finding the radiated waves:

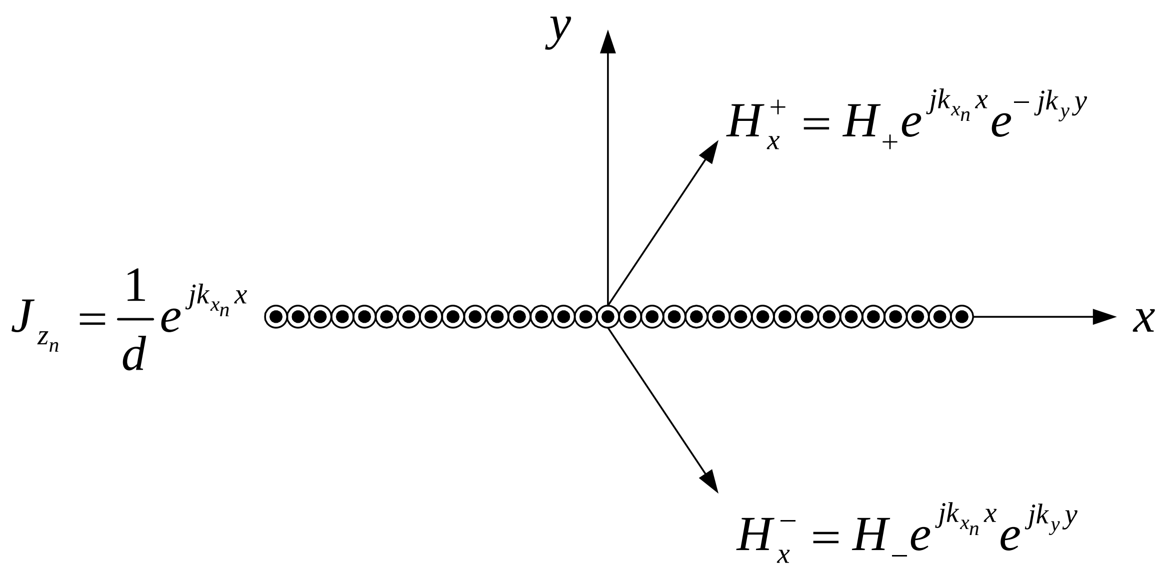


$$J_{z_n} = \frac{1}{d} e^{jk_{x_n}x}$$

$$H_x^+ = H_+ e^{jk_{x_n}x} e^{-jk_y y}$$

$$H_x^- = H_- e^{jk_{x_n}x} e^{jk_y y}$$

- Finding the radiated waves:



- Boundary Condition:

$$\hat{y} \times \left(\vec{H}^+ - \vec{H}^- \right) = \hat{z} J_{z_n}$$

$$H_{x_n} = \begin{cases} -\frac{1}{2d} e^{jk_{x_n}x} e^{-jk_{y_n}y}, & y > 0 \\ \frac{1}{2d} e^{jk_{x_n}x} e^{jk_{y_n}y}, & y < 0 \end{cases}$$

- From the dispersion relationship:

$$k_{y_n} = \sqrt{k_0^2 - k_{x_n}^2}$$

- Normal H can be found via Gauss' law:

$$\nabla \cdot \vec{H}_n = 0$$

$$\vec{k}_n \cdot \vec{H}_n = 0$$

$$k_{x_n} H_{x_n} \mp k_{y_n} H_{y_n}^{\pm} = 0 \qquad \therefore H_{y_n}^{\pm} = \pm \frac{k_{x_n}}{k_{y_n}} H_{x_n}$$

- We can also derive the Magnetic Vector Potential due to the plane wave:

$$\vec{H}_n = \nabla \times \vec{A}_n = \hat{x}\frac{\partial A_{z_n}}{\partial y} - \hat{y}\frac{\partial A_{z_n}}{\partial x} = \hat{x}H_{x_n} + \hat{y}H_{y_n}$$

- Given, H_{x_n}, H_{y_n} from above:

$$A_{z_n} = \frac{1}{j2dk_{y_n}} e^{jk_{x_n}x} e^{-jk_{y_n}|y|}$$

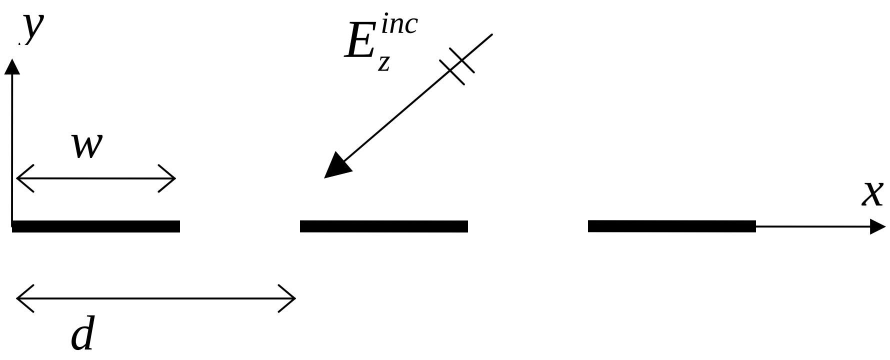
- Finally, the periodic Green function is derived from the summation of all harmonics of the vector potential:

$$\boxed{G_p(x,y) = \sum_{n=-\infty}^{\infty} \frac{e^{jk_{x_n}x} e^{-jk_{y_n}|y|}}{j2dk_{y_n}}}$$

- Observations
  - The spectral domain PGF is a series of Floquet harmonics!
  - Each harmonic has an amplitude proportional to $1/k_{y_n}$
  - Asymptotically, $\lim_{n\to\infty} k_{y_n} \approx -j\dfrac{2\pi n}{d}$
  - $\lim_{n\to\infty} G_{p_n} \approx \dfrac{e^{j\frac{2\pi n}{d}x} e^{-\frac{2\pi n}{d}|y|}}{4\pi n}$
  - Therefore, the series is converging as 1/n, rather than the spatial PGF, which converges as $1/\sqrt{n}$.
  - Note, this is a dramatic improvement, however, it is still quite slow!
  - Fortunately, the convergence can be accelerated
    - This will happen naturally when convolving the PGF with the currents
    - A number of techniques have been introduced to accelerate the convergence even more.

EFIE Solution for the Scattering by a 1D Periodic Strip Array – Solution with the Spectral Green function

- We can return to the MoM solution of the EFIE for the 1D strip array.



- The EFIE is formulated as:

$$E_z^{scat}(x,0)\Big|_{nd \le x \le nd+w} = -E_z^{inc}(x,0)\Big|_{nd \le x \le nd+w}$$

- where,

$$E_z^{scat} = -jk_0\eta_0 A_z$$

$$A_z = \int_0^w J_z(x') G_p(x,y \mid x')\,dx'$$

$$G_p(x,y \mid x') = \sum_{n=-\infty}^{\infty} \frac{e^{jk_{x_n}(x-x')} e^{-jk_{y_n}|y|}}{j2dk_{y_n}}$$

- Again, we assume point basis functions and point test functions
- The EFIE becomes:

$$E_z^{inc}\left(x_j^c\right) = jk_0\eta_0 \sum_{i=1}^{N} \alpha_i \int_{x_i}^{x_{i+1}} \sum_{n=-\infty}^{\infty} \frac{e^{jk_{x_n}\left(x_j^c - x'\right)}}{j2dk_{y_n}} dx'$$

- or,

$$E_z^{inc}\left(x_j^c\right) = \frac{k_0\eta_0}{2d} \sum_{i=1}^{N} \alpha_i \sum_{n=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} \frac{e^{jk_{x_n}\left(x_j^c - x'\right)}}{k_{y_n}} dx'$$

- This leads to a linear system of equations:

$$\overline{e}^{inc} = \overline{\overline{Z}}\,\overline{\alpha}$$

- where,

$$e_j^{inc} = E_z^{inc}\left(x_j^c\right)$$

$$Z_{j,i} = \frac{k_0\eta_0}{2d} \sum_{n=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} \frac{e^{jk_{x_n}\left(x_j^c - x'\right)}}{k_{y_n}} dx'$$

- This integration can be computed analytically!

$$Z_{j,i} = \frac{k_0\eta_0}{2d} \sum_{n=-\infty}^{\infty} \frac{\ell_i \operatorname{sinc}\left(\dfrac{k_{x_n}\ell_i}{2}\right)}{k_{y_n}} e^{jk_{x_n}\left(x_j^c - x_i^c\right)}$$

$$\boxed{Z_{j,i} = \frac{k_0\eta_0}{2d}\sum_{n=-\infty}^{\infty}\frac{\ell_i \operatorname{sinc}\left(\frac{k_{x_n}\ell_i}{2}\right)}{k_{y_n}} e^{jk_{x_n}\left(x_j^c - x_i^c\right)}}$$

- Observation:
  - The sinc function is effectively the Fourier-transform of the pulse basis function.
  - The result is effectively the product of the spectral PGF harmonic and the Fourier harmonic of the pulse basis function
  - Due to the properties of the sinc function, the series now has an asymptotic convergence of $O\left(1/n^2\right)$, which is much improved.
    - What if we used a smoother basis how would this impact the convergence?
  - Due to the product in the spectral domain, the spectral response of the current basis effectively *filters* the spectral Green function.

Computing the Scattered Fields

- The method of moment solution is used to compute the unknown coefficients of the current basis functions.
- Given the solution, we can approximate the currents, and consequently the scattered field:

$$E_z^{scat} = -jk_0\eta_0 A_z = -jk_0\eta_0 \sum_{i=1}^{N} \alpha_i \int_{x_i}^{x_{i+1}} \sum_{n=-\infty}^{\infty} \frac{e^{jk_{x_n}(x-x')} e^{-jk_{y_n}|y|}}{j2dk_{y_n}} dx'$$

$$= -\frac{k_0\eta_o}{2d} \sum_{n=-\infty}^{\infty} \left[\sum_{i=1}^{N} \alpha_i \int_{x_i}^{x_{i+1}} e^{-jk_{x_n}x'} dx' \right] \frac{e^{jk_{x_n}x} e^{-jk_{y_n}|y|}}{k_{y_n}}$$

- where,

$$\int_{x_i}^{x_{i+1}} e^{jk_{x_n}x} dx' = \ell_i e^{-jk_{x_n}x_i^c} \operatorname{sinc}\left(k_{x_n} \frac{\ell_i}{2} \right)$$

- Therefore,

$$E_z^{scat} = \sum_{n=-\infty}^{\infty} \left[-\frac{k_0\eta_o}{2dk_{y_n}} \sum_{i=1}^{N} \alpha_i \ell_i e^{-jk_{x_n}x_i^c} \operatorname{sinc}\left(k_{x_n} \frac{\ell_i}{2} \right) \right] e^{jk_{x_n}x} e^{-jk_{y_n}|y|}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jk_{x_n}x} e^{-jk_{y_n}|y|}$$

- The scattered electric field is thus expanded via a series of Floquet harmonics:

$$E_z^{scat} = \sum_{n=-\infty}^{\infty} c_n e^{jk_{x_n}x} e^{-jk_{y_n}|y|}$$

- With amplitudes:

$$c_n = -\frac{k_0\eta_o}{2d}\sum_{i=1}^{N}\alpha_i \frac{\ell_i e^{-jk_{x_n}x_i^c}}{k_{y_n}} \operatorname{sinc}\left(k_{x_n}\frac{\ell_i}{2}\right)$$

- And with:

$$k_{x_n} = k_o\left(\cos\phi^{inc} + \frac{n}{d/\lambda_0}\right), \quad k_{y_n} = k_0\sqrt{1-\left(\cos\phi^{inc} + \frac{n}{d/\lambda_0}\right)^2}$$

- Therefore, the scattered field are the Floquet harmonics expected, and c_n determine the amplitude of each mode.
- Note that each mode carries real power away from the grating at the Floquet angles determined by ϕ^{inc} and d/λ_0
- Evanescent modes have purely imaginary power along the *y*-direction

Power Spectrum and Conservation of Power

- Only propagating modes will carry power away from the plane of the strip - that is along the normal projection.
- Compute the power density relative to the normal direction in the back scatter region (y > 0) for *propagating modes:*

$$P_{y_n}^{b} = \frac{1}{2}\mathrm{Re}\left(E_{z_n} H_{x_n}^{*}\right) = \frac{1}{2}\mathrm{Re}\left(E_{z_n} \cdot \left(\frac{-1}{j\omega\mu_o}\frac{\partial}{\partial y} E_{z_n} \right)^{*} \right)$$

$$= \frac{1}{2}\mathrm{Re}\left(c_n e^{jk_{x_n}x} e^{-jk_{y_n}y} \cdot \left(\frac{k_{y_n}}{k_o\eta_o} c_n^{*} e^{-jk_{x_n}x} e^{+jk_{y_n}y} \right)\right) = \frac{1}{2}\frac{\left|c_n\right|^2}{\eta_o}\sin\left(\phi_n\right)$$

- where $k_{y_n} = k_o \sin\left(\phi_n\right)$, and ϕ_n is the angle off the positive *x*-axis.
- In the forward scattering region for $n \neq 0$ *propagating modes*:

$$P_{y_n}^{f} = \frac{1}{2}\mathrm{Re}\left(E_{z_n} H_{x_n}^{*}\right) = \frac{1}{2}\mathrm{Re}\left(E_{z_n} \cdot \left(\frac{-1}{j\omega\mu_o}\frac{\partial}{\partial y} E_{z_n} \right)^{*} \right)$$

$$= \frac{1}{2}\mathrm{Re}\left(c_n e^{jk_{x_n}x} e^{jk_{y_n}y} \cdot \left(\frac{-k_{y_n}}{k_o\eta_o} c_n^{*} e^{-jk_{x_n}x} e^{-jk_{y_n}y} \right)\right) = -\frac{1}{2}\frac{\left|c_n\right|^2}{\eta_o}\sin\left(\phi_n\right)$$

- Note that the forward scattered power is negative, reflecting the power flow along the negative y-direction. When $n = 0$ the diffracted field (or transmitted field) is actually a superposition of the incident wave and the Floquet mode:

$$P_{y0}^{f} = \frac{1}{2}\text{Re}\left(E_{z0}^{tot} H_{xp}^{*}\right) = \frac{1}{2}\text{Re}\left(E_{z0}^{tot} \cdot \left(\frac{-1}{j\omega\mu_o} \frac{\partial}{\partial y} E_{z0}^{tot} \right)^{*} \right)$$

$$= \frac{1}{2}\text{Re}\left(\left(c_0 + E_o\right) e^{jk_x^i x} e^{+jk_y^i y} \cdot \left(\frac{-k_y^i}{k_o \eta_o} \left(c_0 + E_o\right)^{*} e^{-jk_x^i x} e^{-jk_y^i y} \right)\right) = -\frac{1}{2}\frac{\left|c_0 + E_o\right|^2}{\eta_o}\sin\left(\phi^i\right)$$

- To understand this, consider the case where the periodic structure reduces to a plane. Both the reflected and the transmitted Floquet harmonics have an amplitude of -1. However, the transmitted field is a superposition of the Floquet mode and the incident field and consequently has an amplitude of 0.

- The incident power is simply

$$P_y^{inc} = \frac{1}{2}\text{Re}\left(E_z^{inc} H_x^{inc^*}\right) = -\frac{1}{2}\frac{\left|E_o\right|^2}{\eta_o}\sin\left(\phi^i\right)$$

- A power check simply requires that the sum of the powers carried in the Floquet modes should sum to the incident power. Or, more appropriately:

$$\frac{\sum_{n=-P_1}^{P_2} P_{y_n}^f - \sum_{n=-P_1}^{P_2} P_{y_n}^b}{P_y^{inc}} = 1$$

- where P_1 and P_2 reflect the bounds of the propagating modes, and the minus sign accounts for the direction of power flow.
  - Note that since P_y^{inc} is negative, we the quotient above will be positive
- Note that a power check is a necessary, but not sufficient condition, that the solution is correct.

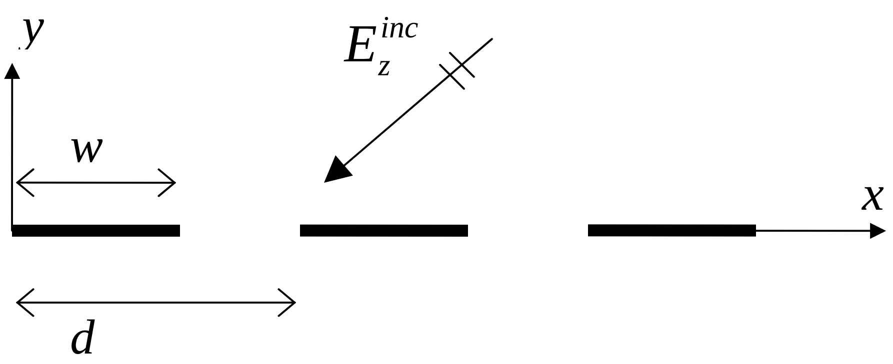
- We can also define the normalized powers of each mode:

$$\tilde{P}_{y_n}^f = \frac{P_{y_n}^f}{P_y^i}, \quad \tilde{P}_{y_n}^b = \frac{P_{y_n}^b}{P_y^i}$$

- Then, for power conservation:

$$\sum_{n=-P_1}^{P_2} \tilde{P}_{y_n}^f - \sum_{n=-P_1}^{P_2} \tilde{P}_{y_n}^b = 1$$

Examples



$$d = 0.8\lambda_0, \quad w = 0.4\lambda_0, \quad \phi^{inc} = 30°$$

50 segments (way over sampled!)

Refraction angles (off –ve x-axis):

$$\phi_0 = 30°$$

$$\phi_{-1} = -67.42°$$

Floquet Amplitudes:

$$e_{-1} = 0.14511 - 0.2285\,j$$

$$e_0 = -0.79076 + 0.18673\,j$$

Powers:

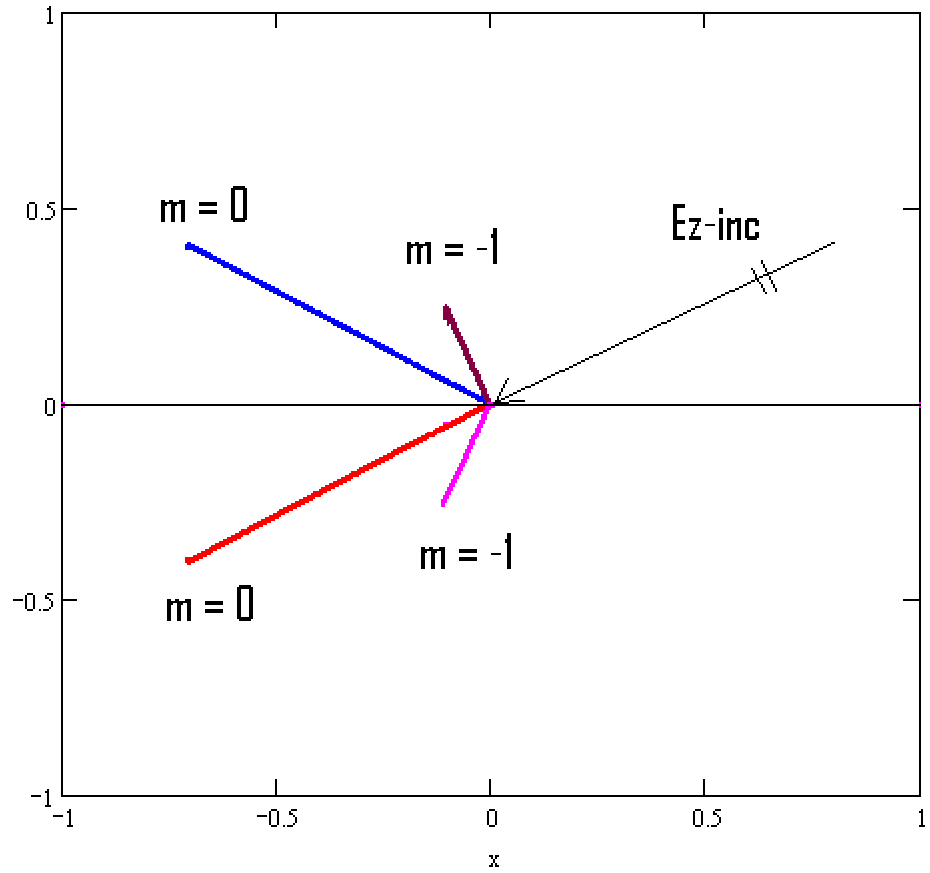
$$P_y^{inc} = -6.63605\times10^{-4}\ \text{W/m}^2$$

$$P_{y_{-1}}^{b} = 8.66667\times10^{-5}\ \text{W/m}^2$$

$$P_{y_{-1}}^{f} = -8.66667\times10^{-5}\ \text{W/m}^2$$

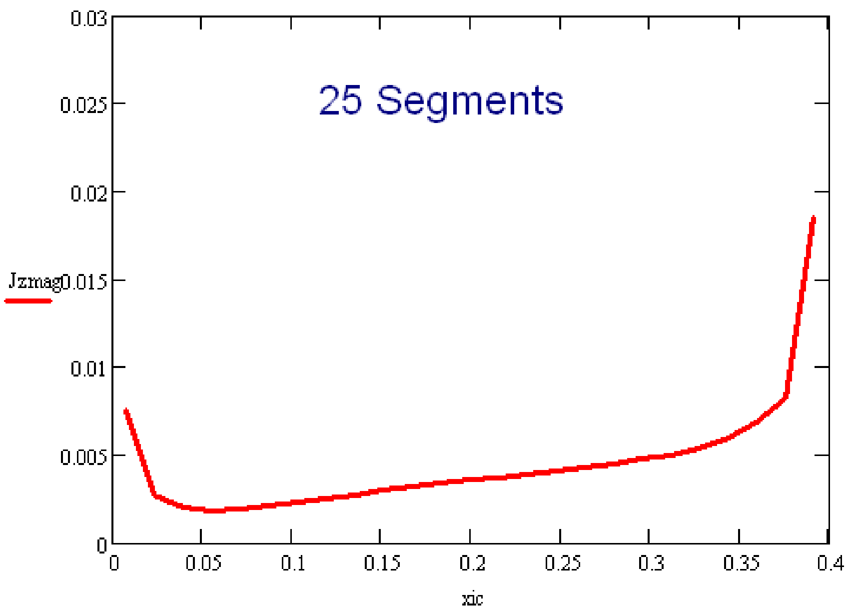
$$P_{y_0}^{b} = 4.38091\times10^{-4}\ \text{W/m}^2$$

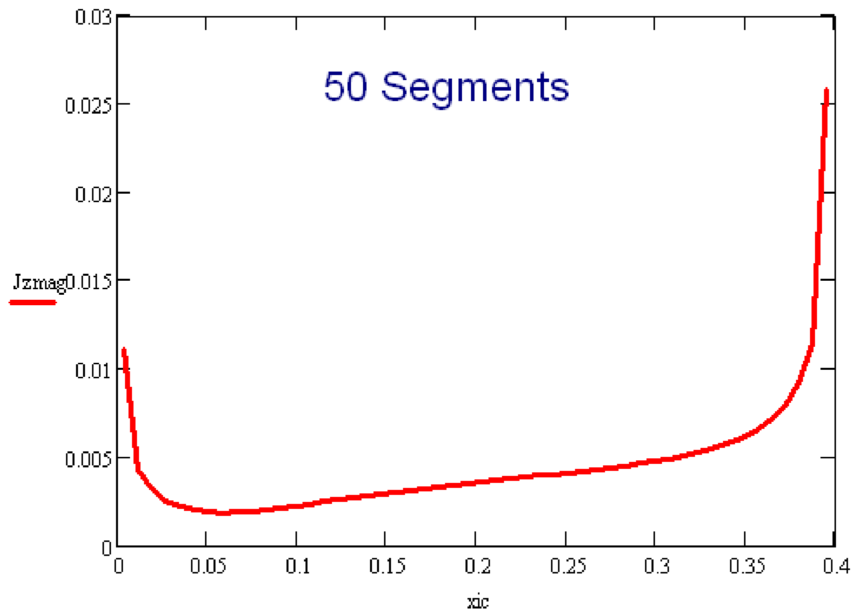
$$P_{y_0}^{f} = -5.21917\times10^{-5}\ \text{W/m}^2$$

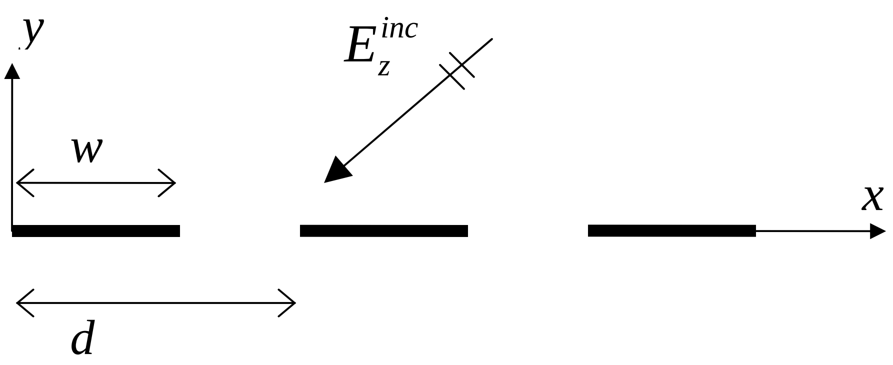


Power check: $\left(\sum_{n=-P_1}^{P_2} P_{y_n}^{f} - \sum_{n=-P_1}^{P_2} P_{y_n}^{b} \right) / P_y^{inc} = 1.00002$

Magnitude of the Current Distribution

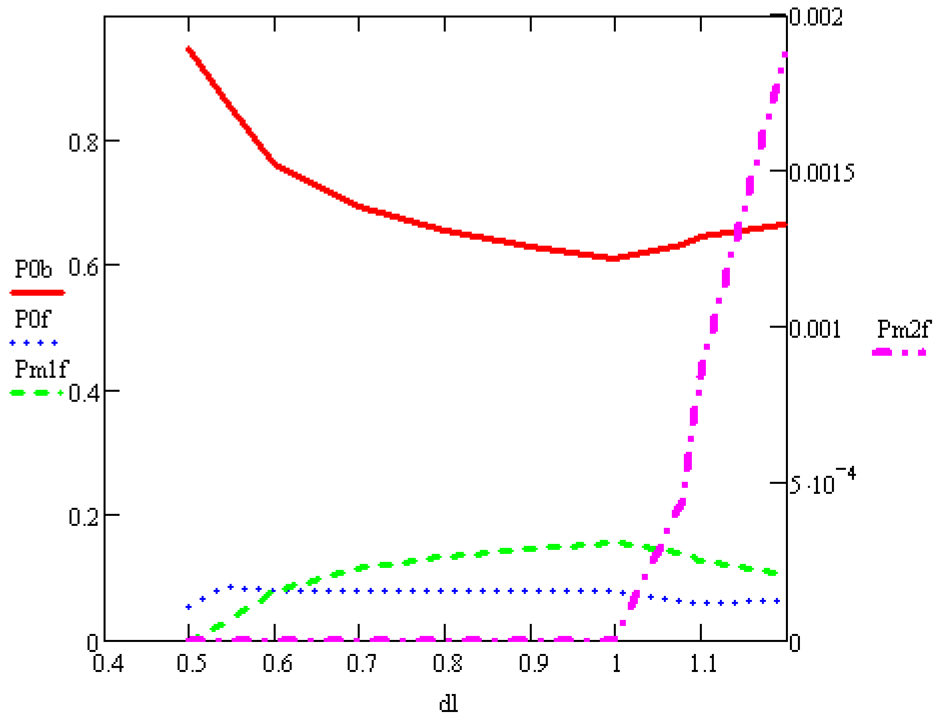






$$d = 0.5 - 1.2\,\lambda_0, \quad w = d/2, \;\; \phi^{inc} = 30°, \text{ 25 segments}$$

$$\frac{P_{y_0}^{b}}{P_y^i}, \frac{P_{y_0}^{f}}{P_y^i}, \frac{P_{y_{-1}}^{f/b}}{P_y^i}$$



$$\frac{P_{y_{-2}}^{f/b}}{P_y^i}$$