Appendix G
Cascaded Scattering Matrices

A split-T Power divider in microstrip or any other transmission line technology may be analysed as consisting of a basic power divider and two impedance transformers. The basic power divider is a transmission line of desired characteristic impedance that splits into two transmission lines of zero length with characteristic impedances determined by the desired power ratio of the two outputs. The impedance transformers are transmission lines of quarter lambda length at the centre frequency that transform these impedance levels to the desired characteristic impedance. The basic power divider may be described by a 3×3 scattering matrix, each impedance transformer may be described by a 2×2 scattering matrix. The overall scattering matrix may be described by a 3×3 scattering matrix.

The cascading process of the basic power divider and the two impedance transformers is schematically shown in figure G.1.

The scattering matrix of the basic power divider is denoted S, the scattering matrices of the two impedance transformers are denoted S', respectively S''.

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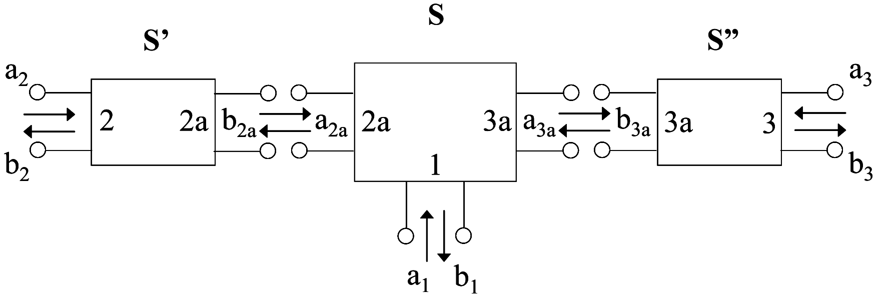

Fig. G.1 Cascading of three-port and two two-ports.

The complex amplitudes of outgoing waves and incoming waves of the basic power divider are related through, see figure G.1

$$b_1 = S_{11}a_1 + S_{12}a_{2a} + S_{13}a_{3a}, \tag{G.1}$$
$$b_{2a} = S_{21}a_1 + S_{22}a_{2a} + S_{23}a_{3a}, \tag{G.2}$$
$$b_{3a} = S_{31}a_1 + S_{32}a_{2a} + S_{33}a_{3a}. \tag{G.3}$$

The complex amplitudes of outgoing waves and incoming waves of the impedance transformer at the left of figure G.1 are related through

$$a_{2a} = S'_{11}b_{2a} + S'_{12}a_2, \tag{G.4}$$
$$b_2 = S'_{21}b_{2a} + S'_{22}a_2, \tag{G.5}$$

and for the impedance transformer at the right of the figure, we find

$$a_{3a} = S''_{11}b_{3a} + S''_{12}a_3, \tag{G.6}$$
$$b_3 = S''_{21}b_{3a} + S''_{22}a_3. \tag{G.7}$$

Substitution of equations (G.4) and (G.6) into equations (G.1), (G.2) and (G.3) results in

$$b_1 = S_{11}a_1 + S_{12}S'_{11}b_{2a} + S_{12}S'_{12}a_2 + S_{13}S''_{11}b_{3a} + S_{13}S''_{12}a_3, \tag{G.8}$$
$$b_{2A} = S_{21}a_1 + S_{22}S'_{11}b_{2a} + S_{22}S'_{12}a_2 + S_{23}S''_{11}b_{3a} + S_{23}S''_{12}a_3, \tag{G.9}$$
$$b_{3A} = S_{31}a_1 + S_{32}S'_{11}b_{2a} + S_{32}S'_{12}a_2 + S_{33}S''_{11}b_{3a} + S_{33}S''_{12}a_3. \tag{G.10}$$

Rearranging the terms in equation (G.10) leads to

$$b_{3a} = \frac{S_{31}}{1 - S_{33}S''_{11}}a_1 + \frac{S_{32}S'_{12}}{1 - S_{33}S''_{11}}a_2 + \frac{S_{33}S'_{12}}{1 - S_{33}S''_{11}}a_3 + \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}b_{2a}. \tag{G.11}$$

Substitution of equation (G.11) in equation (G.9) results in an expression for b_{2a} in terms of a_1, a_2 and a_3

$$\begin{aligned} b_{2a} = {} & \frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} a_1 + \\ & \frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} a_2 + \\ & \frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} a_3. \end{aligned} \tag{G.12}$$

Then, upon substitution of equation (G.12) into equation (G.11) we find for b_{3a}

$$\begin{aligned} b_{3a} = {} & \left[\frac{S_{31}}{(1 - S_{33}S''_{11})} + \right. \\ & \left. \frac{S_{32}S'_{11}}{(1 - S_{33}S''_{11})} \frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} \right] a_1 + \\ & \left[\frac{S_{32}S'_{12}}{(1 - S_{33}S''_{11})} + \right. \\ & \left. \frac{S_{32}S'_{11}}{(1 - S_{33}S''_{11})} \frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} \right] a_2 + \\ & \left[\frac{S_{33}S''_{12}}{(1 - S_{33}S''_{11})} + \right. \\ & \left. \frac{S_{32}S'_{11}}{(1 - S_{33}S''_{11})} \frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} \right] a_3. \end{aligned} \tag{G.13}$$

Substitution of equations (G.12) and (G.13) in, respectively, equations (G.8), (G.5) and (G.7), bearing in mind that the scattering matrix of the complete cascaded system is given by

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} S^T_{11} & S^T_{12} & S^T_{13} \\ S^T_{21} & S^T_{22} & S^T_{23} \\ S^T_{31} & S^T_{32} & S^T_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{G.14}$$

finally yields for the scattering coefficients

$$\begin{aligned} S^T_{11} = {} & S_{11} + S_{12}S'_{11} \frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} + \\ & S_{13}S''_{11} \left[\frac{S_{31}}{1 - S_{33}S''_{11}} + \right. \\ & \left. \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}} \frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} \right], \end{aligned} \tag{G.15}$$

$$S_{12}^{T} = S_{12}S'_{12} + S_{12}S'_{11}\frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} + \\ S_{13}S''_{11}\left[\frac{S_{32}S'_{12}}{1 - S_{33}S''_{11}} + \right. \\ \left. \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}\frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S\text{"}_{11}) - S_{23}S''_{11}S_{32}S'_{11}}\right], \qquad \text{(G.16)}$$

$$S_{13}^{T} = S_{13}S''_{12} + S_{12}S'_{11}\frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}} + \\ S_{13}S''_{11}\left[\frac{S_{33}S''_{12}}{1 - S_{33}S''_{11}} + \right. \\ \left. \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}\frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}\right], \qquad \text{(G.17)}$$

$$S_{21}^{T} = S'_{21}\frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}, \qquad \text{(G.18)}$$

$$S_{22}^{T} = S'_{22} + S'_{21}\frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}, \qquad \text{(G.19)}$$

$$S_{23}^{T} = S'_{21}\frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}, \qquad \text{(G.20)}$$

$$S_{31}^{T}=S''_{21}\left[\frac{S_{31}}{1 - S_{33}S''_{11}} + \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}\frac{S_{21}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{31}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}\right], \qquad \text{(G.21)}$$

$$S_{32}^{T}=S''_{21}\left[\frac{S_{32}S'_{12}}{1 - S_{33}S''_{11}} + \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}\frac{S_{22}S'_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{32}S'_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}\right], \qquad \text{(G.22)}$$

$$S_{33}^{T} = S''_{22} + S''_{21}\left[\frac{S_{33}S''_{12}}{1 - S_{33}S''_{11}} + \right. \\ \left. \frac{S_{32}S'_{11}}{1 - S_{33}S''_{11}}\frac{S_{23}S''_{12}(1 - S_{33}S''_{11}) + S_{23}S''_{11}S_{33}S''_{12}}{(1 - S_{22}S'_{11})(1 - S_{33}S''_{11}) - S_{23}S''_{11}S_{32}S'_{11}}\right]. \qquad \text{(G.23)}$$

Grouping common terms results in the following - easier to handle - equations for the elements of the split-T power divider scattering matrix

$$S_{11}^T = S_{11} + S_{31}G + FC, \tag{G.24}$$
$$S_{12}^T = S_{12}S'_{12} + S_{32}S'_{12}G + FD, \tag{G.25}$$
$$S_{13}^T = S_{13}S''_{12} + S_{33}S\text{"}_{12}G + FE, \tag{G.26}$$
$$S_{21}^T = S'_{21}C, \tag{G.27}$$
$$S_{22}^T = S'_{22} + S'_{21}D, \tag{G.28}$$
$$S_{23}^T = S'_{21}E, \tag{G.29}$$
$$S_{31}^T = S''_{21}\left[\frac{S_{31}}{A} + HC\right], \tag{G.30}$$
$$S_{32}^T = S''_{21}\left[\frac{S_{32}S'_{12}}{A} + HD\right], \tag{G.31}$$
$$S_{33}^T = S''_{22} + S''_{21}\left[\frac{S_{33}S''_{12}}{A} + HE\right], \tag{G.32}$$

where

$$A = 1 - S_{33}S''_{11}, \tag{G.33}$$
$$B = A(1 - S_{22}S'_{11}) - S_{23}S''_{11}S_{32}S'_{11}, \tag{G.34}$$
$$C = \frac{S_{21}A + S_{23}S''_{11}S_{31}}{B}, \tag{G.35}$$
$$D = \frac{S_{22}S'_{12}A + S_{23}S''_{11}S_{32}S'_{12}}{B}, \tag{G.36}$$
$$E = \frac{S_{23}S''_{12}A + S_{23}S''_{11}S_{33}S''_{12}}{B}, \tag{G.37}$$
$$F = S_{12}S'_{11} + \frac{S_{13}S''_{11}S_{32}S'_{11}}{A}, \tag{G.38}$$
$$G = \frac{S_{13}S''_{11}}{A}, \tag{G.39}$$
$$H = \frac{S_{32}S'_{11}}{A}. \tag{G.40}$$