Conventional Beamforming Algorithms

Dr. John McDonough

Spoken Language Systems
Saarland University

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Introduction

- In this lecture, we consider the conventional adaptive beamforming algorithms.
- These algorithms are based on the notion of minimizing the output power of the array subject to a distortionless constraint.
- We will compare the performance of the adaptive beamforming algorithms to the simple delay-and-sum design.
- We will also discuss how beamforming performance can be improved through the addition of a postfilter.

Coverage: Wölfel and McDonough, Sections 13.2 and 13.3, Appendix

Beam Pattern of Delay-and-Sum Beamformer

- Recall that in u-space, the beam pattern can be specified as

$$B_u(u) = \frac{1}{N}\,\frac{\sin\left(\frac{\pi N d}{\lambda}u\right)}{\sin\left(\frac{\pi d}{\lambda}u\right)}, \quad \text{for } -1 \le u \le 1.$$

- Nulls occur when the numerator of $B_u(u)$ is zero and the denominator is non-zero.
- Note that

$$\sin\left(\frac{\pi N d}{\lambda}u\right) = 0,$$

  when

$$\frac{\pi N d}{\lambda}u = m\pi, \quad \text{for } m = 1, 2, \ldots.$$

Null-to-Null Beamwidth

- The nulls occur when both

$$u = m\frac{\lambda}{Nd} \text{ for } m = 1, 2, \ldots,$$
$$u \neq m\frac{\lambda}{d} \text{ for } m = 1, 2, \ldots.$$

- Hence, the first null occurs at $u = \lambda / Nd$ and the *null-to-null beamwidth* $\mathrm{BW}_{\mathrm{NN}}$ is

$$\Delta u_2 = 2\frac{\lambda}{Nd}.$$

Complex Gradients

- Define the complex vector

$$\mathbf{z} = \mathbf{x} + j\mathbf{y}.$$

- Also define the complex gradient operators

$$\nabla_{\mathbf{z}} = \begin{bmatrix} \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \cdots & \frac{\partial}{\partial z_N} \end{bmatrix}^T,$$

$$\nabla_{\mathbf{z}^H} = \begin{bmatrix} \frac{\partial}{\partial z_1^*} & \frac{\partial}{\partial z_2^*} & \cdots & \frac{\partial}{\partial z_N^*} \end{bmatrix}.$$

Stationary Points of Functions of a Complex Vector

- Let
$$f(\mathbf{z}) = f(\mathbf{x}, \mathbf{y}) = g(\mathbf{z}, \mathbf{z}^H),$$
where $g(\mathbf{z}, \mathbf{z}^*)$ is a real-valued function of $\mathbf{z}$ and $\mathbf{z}^*$, which is analytic with respect to $\mathbf{z}$ and $\mathbf{z}^*$ independently.
- Then either
$$\nabla_{\mathbf{z}}\, \mathbf{g}(\mathbf{z}, \mathbf{z}^H) = \mathbf{0}$$
where $\mathbf{z}^H$ is treated as a constant, or
$$\nabla_{\mathbf{z}^H}\, \mathbf{g}(\mathbf{z}, \mathbf{z}^H) = \mathbf{0}$$
where $\mathbf{z}$ is treated as a constant, is a necessary and sufficient condition for a stationary point of $f(\mathbf{z})$.

Method of Undetermined Lagrange Multipliers

- Consider the *constrained optimization problem*:

$$\text{minimize } \mathbf{w}^H \mathbf{S} \mathbf{w},$$
$$\text{subject to } \mathbf{w}^H \mathbf{c} = g.$$

- To apply the *method of undetermined Lagrange multipliers*, define

$$\begin{aligned} J(\mathbf{w}, \mathbf{w}^H) &= \mathbf{w}^H \mathbf{S} \mathbf{w} + 2\mathrm{Re}\left[\lambda \left(\mathbf{w}^H \mathbf{c} - g\right)\right] \\ &= \mathbf{w}^H \mathbf{S} \mathbf{w} + \lambda \left(\mathbf{w}^H \mathbf{c} - g\right) + \lambda^* \left(\mathbf{c}^H \mathbf{w} - g\right). \end{aligned}$$

- Now take the derivative with respect to $\mathbf{w}^H$ and equate to zero, to obtain

$$\mathbf{S} \mathbf{w} + \lambda \mathbf{c} = 0,$$

or

$$\mathbf{w} = -\lambda \mathbf{S}^{-1} \mathbf{c}.$$

Undetermined Lagrange Multipliers (cont'd.)

- Applying the constraint we find,

$$\mathbf{w}^H\mathbf{c} = -\lambda\mathbf{c}^H\mathbf{S}^{-1}\mathbf{c} = g,$$

or

$$\lambda = \frac{-g}{\mathbf{c}^H\mathbf{S}^{-1}\mathbf{c}}.$$

- Then, the final solution is

$$\mathbf{w}^H = \frac{g\,\mathbf{c}^H\mathbf{S}^{-1}}{\mathbf{c}^H\mathbf{S}^{-1}\mathbf{c}}.$$

Matrix Inversion Lemma

- Consider the matrices **A**, **B**, **C**, and **D** where **A** is $N \times N$, **B** is $N \times M$, **C** is $M \times M$ and **D** is $M \times N$.
- The *matrix inversion lemma* states

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{DA}^{-1}\mathbf{B} + \mathbf{C}^{-1}\right)^{-1}\mathbf{DA}^{-1}.$$

- An important special case is where **B** is an $N \times 1$ column vector **x**, **C** is a scalar c, and $\mathbf{D} = \mathbf{x}^H$:

$$\left(\mathbf{A} + c\mathbf{x}\mathbf{x}^H\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{x}\mathbf{x}^H\mathbf{A}^{-1}}{c^{-1} + \mathbf{x}^H\mathbf{A}^{-1}\mathbf{x}}.$$

- *Woodbury's identity* is obtained by setting $c = 1$, such that

$$\left(\mathbf{A} + \mathbf{x}\mathbf{x}^H\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{x}\mathbf{x}^H\mathbf{A}^{-1}}{1 + \mathbf{x}^H\mathbf{A}^{-1}\mathbf{x}}.$$

Matrix Inversion Lemma (cont'd.)

Several other useful relations follow from the matrix inversion lemma:

$$\left(\mathbf{A}^{-1} + \mathbf{B}^H \mathbf{C}^{-1} \mathbf{B}\right)^{-1} = \mathbf{A} - \mathbf{A}\mathbf{B}^H \left(\mathbf{B}\mathbf{A}\mathbf{B}^H + \mathbf{C}\right)^{-1} \mathbf{B}\mathbf{A},$$

$$\left(\mathbf{A}^{-1} + \mathbf{B}^H \mathbf{C}^{-1} \mathbf{B}\right)^{-1} \mathbf{B}^H \mathbf{C}^{-1} = \mathbf{A}\mathbf{B}^H \left(\mathbf{B}\mathbf{A}\mathbf{B}^H + \mathbf{C}\right)^{-1},$$

$$\mathbf{C}^{-1} - \left(\mathbf{B}\mathbf{A}\mathbf{B}^H + \mathbf{C}\right)^{-1} = \mathbf{C}^{-1}\mathbf{B} \left(\mathbf{A}^{-1} + \mathbf{B}^H \mathbf{C}^{-1} \mathbf{B}\right)^{-1} \mathbf{B}^H \mathbf{C}^{-1}.$$

Perpendicular Projection

- Consider an N-dimensional vector $\mathbf{x}$.
- Consider also an $N \times M$ matrix $\mathbf{C}$ whose linearly independent columns define an M-dimensional *subspace* of the complete N-dimensional space.
- We wish to find the *perpendicular projection* of $\mathbf{x}$ onto the $\mathbf{C}$ subspace.
- The projection can be expressed as $\mathbf{Cy}$ where the M-dimensional $\mathbf{y}$ minimizes

$$\begin{aligned}\|\mathbf{x} - \mathbf{Cy}\|^2 &= (\mathbf{x} - \mathbf{Cy})^H(\mathbf{x} - \mathbf{Cy}) \\ &= \mathbf{x}^H\mathbf{x} - \mathbf{y}^H\mathbf{C}^H\mathbf{x} - \mathbf{x}^H\mathbf{Cy} + \mathbf{y}^H\mathbf{C}^H\mathbf{Cy}\end{aligned}$$

Perpendicular Projection Operator

- Taking the gradient with respect to $\mathbf{y}^H$ and equating to zero, we find
$$-\mathbf{C}^H\mathbf{x} + \mathbf{C}^H\mathbf{C}\mathbf{y} = \mathbf{0}$$
or
$$\hat{\mathbf{y}} = (\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\mathbf{x}$$
- The inverse must exist, because the columns of $\mathbf{C}$ are linearly independent.
- Hence, the desired projection is
$$\mathbf{x}_\mathbf{C} = \mathbf{C}\hat{\mathbf{y}} = \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\mathbf{x}$$
- The *perpendicular projection operator* is then
$$\boxed{\mathbf{P}_\mathbf{C} = \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H}$$

Array Gain

- We will in all cases take *word error rate* (WER) as the most important measure of system performance.
- It is useful, however, to have other performance measures intended a single component.
- *Signal-to-noise ratio* (SNR), a very common metric for signal quality, is the ratio of signal power to noise power.
- *Array gain* is a measure of how much improvement in SNR is achieved by a sensor array.
- Array gain is the ratio of SNR at the output of the array to that at input of any given sensor.

Snapshot Model

- Let $\mathbf{X}(\omega) \in \mathbb{C}^N$ denote a subband domain snapshot, a vector of N complex subband samples, one per microphone, obtained from

$$\mathbf{X}(\omega) = \mathbf{F}(\omega) + \mathbf{N}(\omega), \tag{1}$$

  where $\mathbf{F}(\omega)$ and $\mathbf{N}(\omega)$ denote the subband-domain snapshot of the desired signal and noise or interference.
- We will assume that $\mathbf{F}(\omega)$ and $\mathbf{N}(\omega)$ are uncorrelated and that the signal vector $\mathbf{F}(\omega)$ can be expressed as

$$\mathbf{F}(\omega) = F(\omega)\, \mathbf{v_k}(\mathbf{k}), \tag{2}$$

  where $\mathbf{v_k}(\mathbf{k})$ is the array manifold vector.

Second Order Statistics

- We now introduce the notation necessary for specifying the second order statistics of random variables and vectors.
- In general, for some complex scalar random variable $Y(\omega)$, we will define

$$\Sigma_Y(\omega) \triangleq \mathcal{E}\{|Y(\omega)|^2\}.$$

  Similarly, for a complex random vector $\mathbf{X}(\omega)$, let us define the *spatial spectral matrix* as

$$\mathbf{\Sigma}_{\mathbf{X}}(\omega) \triangleq \mathcal{E}\{\mathbf{X}(\omega)\mathbf{X}^H(\omega)\}.$$

Signal and Noise Components

- Assume that the component of the desired signal reaching each microphone is $F(\omega)$ and the component of the noise or interference reaching each sensor is $N(\omega)$.
- This implies that the SNR at the input of the array is

$$\mathrm{SNR}_{\mathrm{in}}(\omega) \triangleq \frac{\Sigma_F(\omega)}{\Sigma_N(\omega)}. \tag{3}$$

- In the frequency domain, the output of the beamformer is

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \mathbf{H}^T(\omega)\, \mathbf{X}(\omega). \tag{4}$$

- Defining $\mathbf{w}^H(\omega) = \mathbf{H}^T(\omega)$ enables (4) to be rewritten as

$$Y(\omega) = \mathbf{w}^H(\omega)\, \mathbf{X}(\omega) = Y_F(\omega) + Y_N(\omega), \tag{5}$$

where $Y_F(\omega) = \mathbf{w}^H(\omega)\, \mathbf{F}(\omega)$ and $Y_N(\omega) = \mathbf{w}^H(\omega)\, \mathbf{N}(\omega)$ are the signal and noise components in the beamformer output.

Beamformer Output

- When the delay-and-sum beamformer (DSB) is steered to wavenumber $\mathbf{k} = \mathbf{k}_T$, the sensor weights become

$$\mathbf{w}^H = \frac{1}{N}\, \mathbf{v_k}^H(\mathbf{k}_T). \tag{6}$$

- The variance of the output of the beamformer can be calculated according to

$$\Sigma_Y(\omega) = \mathcal{E}\{|\,Y(\omega)|^2\} = \Sigma_{Y_F}(\omega) + \Sigma_{Y_N}(\omega), \tag{7}$$

  where

$$\Sigma_{Y_F}(\omega) = \mathbf{w}^H(\omega)\, \boldsymbol{\Sigma}_\mathbf{F}(\omega)\, \mathbf{w}(\omega), \tag{8}$$

  is the signal component of the beamformer output, and

$$\Sigma_{Y_N}(\omega) = \mathbf{w}^H(\omega)\, \boldsymbol{\Sigma}_\mathbf{N}(\omega)\, \mathbf{w}(\omega), \tag{9}$$

  is the noise component.

- Equation (9) follows directly from the assumption that $\mathbf{F}(\omega)$ and $\mathbf{N}(\omega)$ are uncorrelated.

Signal Component in Beamformer Output

- Expressing the snapshot of the desired signal once more as in (2), we find that the spatial spectral matrix $\mathbf{F}(\omega)$ of the desired signal can be written as

$$\boldsymbol{\Sigma}_{\mathbf{F}}(\omega) = \Sigma_F(\omega)\,\mathbf{v}_{\mathbf{k}}(\mathbf{k}_s)\,\mathbf{v}_{\mathbf{k}}^H(\mathbf{k}_s), \tag{10}$$

  where $\Sigma_F(\omega) = \{|F(\omega)|^2\}$.

- Substituting (10) into (8), we can calculate the output signal spectrum as

$$\Sigma_{Y_F}(\omega) = \mathbf{w}^H(\omega)\,\mathbf{v}_{\mathbf{k}}(\mathbf{k}_s)\,\Sigma_F(\omega)\,\mathbf{v}_{\mathbf{k}}^H(\mathbf{k}_s)\,\mathbf{w}(\omega) = \Sigma_F(\omega),$$

  where the final equality follows from the definition (6) of the DSB.

Array Gain of Delay-and-Sum Beamformer

- Substituting (6) into (9) the noise at the DSB output is

$$\Sigma_{Y_N}(\omega) = \frac{1}{N^2}\mathbf{v}^H(\mathbf{k}_s)\boldsymbol{\rho}_{\mathbf{N}}(\omega)\mathbf{v}(\mathbf{k}_s)\Sigma_N(\omega), \tag{11}$$

where the *normalized spatial spectral matrix* $\rho_N(\omega)$ is

$$\boldsymbol{\Sigma}_{\mathbf{N}}(\omega) \triangleq \Sigma_N(\omega)\,\boldsymbol{\rho}_{\mathbf{N}}(\omega). \tag{12}$$

- Hence, the SNR at the output of the beamformer is given by

$$\mathrm{SNR}_{\mathrm{out}}(\omega) \triangleq \frac{\Sigma_{Y_F}(\omega)}{\Sigma_{Y_N}(\omega)} = \frac{\Sigma_F(\omega)}{\mathbf{w}^H(\omega)\,\boldsymbol{\Sigma}_{\mathbf{N}}(\omega)\mathbf{w}(\omega)}. \tag{13}$$

- Then based on (3) and (13), the array gain of the DSB is

$$A_{\mathrm{dsb}}(\omega, \mathbf{k}_s) = \frac{\Sigma_{Y_F}(\omega)}{\Sigma_{Y_N}(\omega)} \Big/ \frac{\Sigma_F(\omega)}{\Sigma_N(\omega)} = \frac{N^2}{\mathbf{v}^H(\mathbf{k}_s)\,\boldsymbol{\rho}_{\mathbf{N}}(\omega)\,\mathbf{v}(\mathbf{k}_s)}. \tag{14}$$

Distortionless Constraint

- Many adaptive beamforming algorithms impose a *distortionless constraint*.
- For a plane wave arriving along the main response axis $\mathbf{k}_s$

$$Y(\omega) = F(\omega), \tag{15}$$

  where $Y(\omega)$ is the beamformer output, and $F(\omega)$ is the source signal.
- It follows that

$$Y(\omega) = F(\omega)\,\mathbf{w}^H(\omega)\,\mathbf{v}(\mathbf{k}_s) = F(\omega).$$

- Hence, the distortionless constraint can be expressed as

$$\mathbf{w}^H(\omega)\,\mathbf{v}(\mathbf{k}_s) = 1. \tag{16}$$

- Clearly setting $\mathbf{w}^H(\omega) = \frac{1}{N}\mathbf{v}^H(\mathbf{k}_s)$, as is the case for the DSB, will satisfy (16).
- Thus, the DSB satisfies the distortionless constraint.

Snapshot Model

- The noise snapshot model has spatial spectral matrix

$$\boldsymbol{\Sigma}_{\mathbf{N}}(\omega) = E\{\mathbf{N}(\omega)\mathbf{N}^{H}(\omega)\} = \boldsymbol{\Sigma}_{\mathrm{c}}(\omega) + \sigma_{w}^{2}\mathbf{I},$$

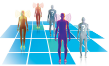
  where $\boldsymbol{\Sigma}_{\mathrm{c}}$ and $\sigma_{\mathrm{w}}^{2}\mathbf{I}$ are the spatially correlated and uncorrelated portions, respectively, of the noise covariance matrix.
- Spatially correlated interference is due to the propagation of some interfering signal through space.
- Uncorrelated noise is due to the self-noise of the sensors.
- The beamformer output will be specified as

$$Y(\omega) = \mathbf{w}^{H}(\omega)\,\mathbf{X}(\omega) = Y_{F}(\omega) + Y_{N}(\omega).$$

- When noise is present, we can write

$$Y(\omega) = F(\omega) + Y_{N}(\omega),$$

  where $Y_{N}(\omega) = \mathbf{w}^{H}(\omega)\mathbf{N}(\omega)$ is the remaining noise component.



Optimization Criterion

- In addition to satisfying the distortionless constraint, we wish also to minimize this output variance, and thereby minimize the influence of the noise.
- To solve the constrained optimization problem, we can apply the method of Lagrange multipliers.
- To wit, we first define the "symmetric" objective function

$$F \triangleq \mathbf{w}^H(\omega)\,\mathbf{\Sigma}_{\mathbf{N}}(\omega)\,\mathbf{w}(\omega) + \lambda[\mathbf{w}^H(\omega)\mathbf{v}(\mathbf{k}_s) - 1] + \lambda^*[\mathbf{v}(\mathbf{k}_s)^H\mathbf{w} - 1], \tag{17}$$

  where λ is a complex Lagrange multiplier, to incorporate the constraint into the objective function.
- Taking the complex gradient with respect to $\mathbf{w}^H$, equating this gradient to zero, and solving yields

$$\mathbf{w}^H_{\text{mvdr}}(\omega) = -\lambda\,\mathbf{v}^H(\mathbf{k}_s)\,\mathbf{\Sigma}_{\mathbf{N}}^{-1}(\omega).$$

Minimum Variance Distortionless Response Beamformer

- Applying now the distortionless constraint (16), we find

$$\lambda = -\left[\mathbf{v}^H(\mathbf{k}_s)\,\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega)\,\mathbf{v}(\mathbf{k}_s)\right]^{-1}.$$

- Thus, the optimal sensor weights are given by

$$\mathbf{w}_o^H(\omega) = \Lambda(\omega)\,\mathbf{v}^H(\mathbf{k}_s)\,\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega) = \mathbf{w}_{\mathrm{mvdr}}^H(\omega), \tag{18}$$

where

$$\Lambda(\omega) \triangleq \left[\mathbf{v}^H(\mathbf{k}_s)\,\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega)\,\mathbf{v}(\mathbf{k}_s)\right]^{-1}. \tag{19}$$

MVDR Schematic

$$\mathbf{X}(\omega) \longrightarrow \boxed{\Lambda(\omega)\mathbf{v}^{H}(\omega : k_s)\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega)} \longrightarrow Y(\omega)$$

- The figure is a schematic of the MVDR beamformer.
- The quantity $\Lambda(\omega)$ is equivalent to the spectral power of the noise component present in $Y(\omega)$, as can be seen from

$$\begin{aligned}
\Sigma_{Y_N}(\omega) &= \mathbf{w}_{\mathrm{mvdr}}^{H}(\omega)\,\boldsymbol{\Sigma}_{\mathbf{N}}(\omega)\,\mathbf{w}_{\mathrm{mvdr}}(\omega) && (20)\\
&= \mathbf{v}^{H}(\mathbf{k}_s)\,\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega)\,\boldsymbol{\Sigma}_{\mathbf{N}}(\omega)\,\boldsymbol{\Sigma}_{\mathbf{N}}^{-1}(\omega)\,\mathbf{v}(\mathbf{k}_s)\cdot\Lambda^{2}(\omega) \\
&= \Lambda(\omega). && (21)
\end{aligned}$$

Advantages of Subband Processing

- The foregoing implies that the sensor weights for each subband are designed independently.
- In particular, the transformation into the subband domain has the effect of a divide and conquer optimization scheme.
- A single optimization problem over MN free parameters, where M is the number of subbands and N is the number of sensors, is converted into M optimization problems, each with N free parameters.
- Each of the M optimization problems is solved independently, which is a direct result of the statistical independence of the subband samples.
- A synthesis filter transforms the beamformed subband samples back into the time domain.

Array Gain of MVDR Beamformer

- As $\mathbf{w}^H_{\text{mvdr}}(\omega)$ satisfies the distortionless constraint, we can write

$$\Sigma_{Y_F}(\omega) = \Sigma_F(\omega),$$

  where $\Sigma_F(\omega)$ is the power spectrum of the desired signal $F(\omega)$ at the input of each sensor.

- Hence, based on (21), the output SNR can be written as $\Sigma_F(\omega)/\Sigma_{Y_N}(\omega) = \Sigma_F(\omega)/\Lambda(\omega)$.
- If we assume the noise spectrum at the input of each sensor is the same, then the input SNR is $\Sigma_F(\omega)/\Sigma_N(\omega)$.
- The array gain can then be expressed as

$$\begin{aligned} A_{\text{mvdr}}(\omega, \mathbf{k}_s) &= \frac{\Sigma_F(\omega)}{\Lambda(\omega)} \bigg/ \frac{\Sigma_F(\omega)}{\Sigma_N(\omega)} = \frac{\Sigma_N(\omega)}{\Lambda(\omega)} \\ &= \Sigma_N(\omega)\, \mathbf{v}^H(\mathbf{k}_s)\, \mathbf{\Sigma}_{\mathbf{N}}^{-1}(\omega)\, \mathbf{v}(\mathbf{k}_s). \end{aligned} \tag{22}$$

Spatially Correlated and Uncorrelated Noise

- Hence, the array gain can be expressed as

$$A_{\mathrm{mvdr}}(\omega, \mathbf{k}_s) = \mathbf{v}^H(\mathbf{k}_s)\, \rho_{\mathbf{N}}^{-1}(\omega)\, \mathbf{v}(\mathbf{k}_s). \tag{23}$$

- If the noises at all sensors are spatially uncorrelated, then $\rho_{\mathbf{N}}(\omega)$ is the identity matrix and the MVDR beamformer reduces to the DSB.
- In this case, the array again is

$$A_{\mathrm{mvdr}}(\omega, \mathbf{k}_s) = A_{\mathrm{dsb}}(\omega, \mathbf{k}_s) = N. \tag{24}$$

In all other cases,

$$A_{\mathrm{mvdr}}(\omega, \mathbf{k}_s) > A_{\mathrm{dsb}}(\omega, \mathbf{k}_s).$$

MVDR Beamformer with Plane Wave Interference

- Consider a desired signal with array manifold vector $\mathbf{v}(\mathbf{k}_s)$ and a single plane-wave interfering signal with manifold vector $\mathbf{v}(\mathbf{k}_1)$.
- In addition, there is uncorrelated sensor noise with power σ_w^2.
- In this case, the spatial spectral matrix $\mathbf{\Sigma_N}(\omega)$ is

$$\mathbf{\Sigma_N}(\omega) = \sigma_w^2\,\mathbf{I} + M_1(\omega)\,\mathbf{v}(\mathbf{k}_1)\,\mathbf{v}^H(\mathbf{k}_1), \tag{25}$$

  where $M_1(\omega)$ is the spectrum of the interfering signal.
- Applying the matrix inversion lemma to (25) provides

$$\mathbf{\Sigma_N}^{-1} = \frac{1}{\sigma_w^2}\left[\mathbf{I} - \frac{M_1}{\sigma_w^2 + N\,M_1}\mathbf{v}_1\mathbf{v}_1^H\right], \tag{26}$$

  where ω and $\mathbf{k}$ have been suppressed, and $\mathbf{v}_1 \triangleq \mathbf{v}(\mathbf{k}_1)$.
- The noise spectrum at each element of the array is then

$$\Sigma_N = \sigma_w^2 + M_1. \tag{27}$$

MVDR with Plane Wave Interference (cont'd.)

- Substituting (26) into (18), we find

$$\mathbf{w}_{\mathrm{mvdr}}^H = \frac{\Lambda}{\sigma_{\mathrm{w}}^2}\,\mathbf{v}_{\mathrm{s}}^H \left[\mathbf{I} - \frac{M_1}{\sigma_{\mathrm{w}}^2 + N\,M_1}\mathbf{v}_1\mathbf{v}_1^H\right]. \tag{28}$$

- The *spatial correlation coefficient* is by definition

$$\rho_{\mathrm{s}1} \triangleq \frac{\mathbf{v}_{\mathrm{s}}^H\mathbf{v}_1}{N}, = B_{\mathrm{dsb}}(\mathbf{k}_1 : \mathbf{k}_{\mathrm{s}}),$$

  where $B_{\mathrm{dsb}}(\mathbf{k}_1 : \mathbf{k}_{\mathrm{s}})$ is the DSB pattern aimed at $\mathbf{k}_{\mathrm{s}}$ and evaluated at $\mathbf{k}_1$.

- With this definition (28) can be rewritten as

$$\mathbf{w}_{\mathrm{mvdr}}^H = \frac{\Lambda}{\sigma_{\mathrm{w}}^2}\left[\mathbf{v}_{\mathrm{s}}^H - \rho_{\mathrm{s}1}\frac{NM_1}{\sigma_{\mathrm{w}}^2 + NM_1}\,\mathbf{v}_1^H\right]. \tag{29}$$

MVDR with Plane Wave Interference (cont'd.)

- The normalizing coefficient (19) then reduces to

$$\Lambda = \left\{ \frac{1}{\sigma_{\mathrm{w}}^2} N \left[1 - \frac{NM_1}{\sigma_{\mathrm{w}}^2 + NM_1} |\rho_{\mathrm{s}1}|^2 \right] \right\}^{-1} . \tag{30}$$

- The upper and lower branches of this MVDR beamformer are conventional beamformers pointing at the desired signal and the interference.
- The necessity of the bottom branch is apparent from:
  - The path labeled $\hat{\mathbf{N}}_1(\omega)$ is the minimum mean-square estimate of the interference plus noise.
  - This noise estimate is scaled by $\rho_{\mathrm{s}1}$ and subtracted from the output of the DSB in the upper path, in order to remove that portion of the noise and interference captured by the upper path.

Schematic: MVDR Beamformer with Plane Wave Interference

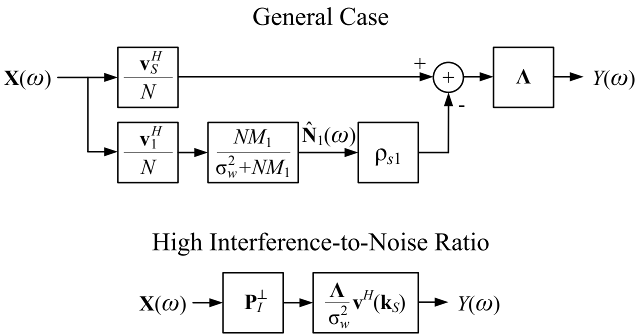


Figure: Optimum MVDR beamformer in the presence of a single interferer.

MVDR with Plane Wave Interference (cont'd.)

- Observe that in the case where $NM_1 \gg \sigma_{\mathrm{w}}^2$, we may rewrite (29) as

$$\mathbf{w}_{\mathrm{mvdr}}^H = \frac{\Lambda}{\sigma_{\mathrm{w}}^2} \mathbf{v}_{\mathrm{s}}^H \mathbf{P}_I^{\perp},$$

  where $\mathbf{P}_I^{\perp} = \mathbf{I} - \mathbf{v}_1 \mathbf{v}_1^H$ is the projection matrix onto the space orthogonal to the interference.
- This case is shown schematically in Figure 1, which indicates that the beamformer is placing a perfect null on the interference.

Limiting Case: Plane Wave Interference

- Substituting (27) and (30) into (22), the array gain of the MVDR beamformer in the presence of plane wave interference is

$$A_{\mathrm{mvdr}} = N(1+\sigma_{\mathrm{I}}^2)\left[\frac{1+N\sigma_{\mathrm{I}}^2(1-|\rho_{\mathrm{s1}}|^2)}{1+N\sigma_{\mathrm{I}}^2}\right],$$

  where the *interference-to-noise ratio* (INR)

$$\sigma_{\mathrm{I}}^2 \triangleq \frac{M_1}{\sigma_{\mathrm{w}}^2}$$

  is the ratio of spatially correlated to uncorrelated noise.

- Beam patterns corresponding to several values of σ_{I}^2 and u_{I}, the direction cosine of the interference, are shown in Figure 2.
- Observe that the suppression of the interference is not perfect when either σ_{I}^2 is verly low, or u_{I} is very small such that the interference moves within the main lobe region of the delay-and-sum beam pattern.

MVDR Beam Patterns

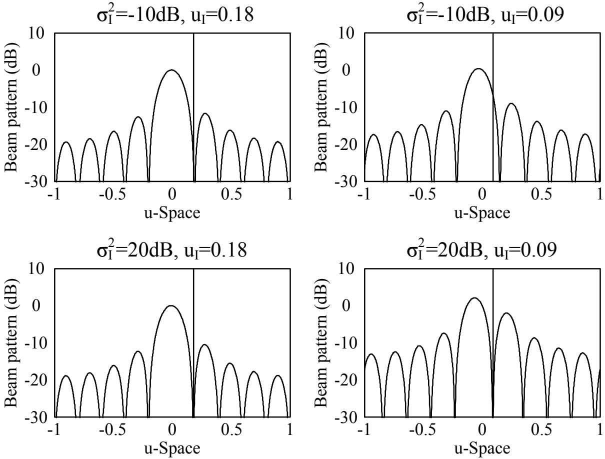


Figure: MVDR beam patterns for plane wave interference.

Performance Curves

- Substituting (25) and (27) into (14), the array gain of the DSB is

$$A_{\text{dsb}}(\omega, \mathbf{k}_{\text{s}}) = \frac{N^2(\sigma_{\text{w}}^2 + M_1)}{\mathbf{v}_{\text{s}}^H(\sigma_{\text{w}}^2 \mathbf{I} + M_1 \mathbf{v}_1 \mathbf{v}_1^H)\mathbf{v}_{\text{s}}} = \frac{N(1 + \sigma_{\text{I}}^2)}{1 + \sigma_{\text{I}}^2 N |\rho_{\text{s}1}|^2}.$$

- From Figure 3 several facts are evident:
  1. When $1 - |\rho_{\text{s}1}|^2$ approaches zero, the array gains of both the DSB and MVDR beamformers drop to zero.
  2. For moderate to high values of σ_{I}^2 and $1 - |\rho_{\text{s}1}|^2 > 0.2$, the MVDR beamformer has a higher array gain than the DSB.
  3. The array gain of the MVDR beamformer increases with σ_{I}^2.
  4. For very low values of σ_{I}^2, the array gain provided by both beamformers approaches 10 dB regardless of $1 - |\rho_{\text{s}1}|^2$. This is to be expected given that the MVDR becomes a DSB for $\boldsymbol{\rho}_{\mathbf{N}}(\omega) = \mathbf{I}$.
  5. That the array gain for both beamformers should be 10 dB is evident from (24).

Array Gains of MVDR and DSB

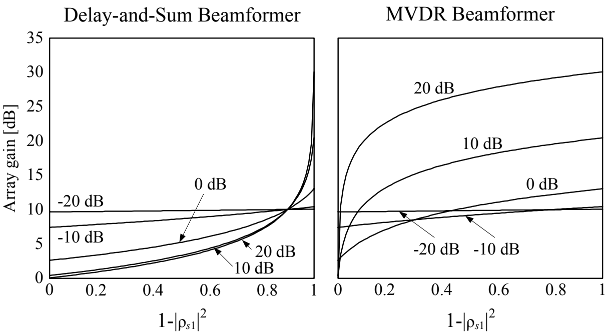


Figure: Array gains for conventional and MVDR beamformers as a function of $(1 - |\rho_{s1}|^2)$ for a 10-element array. The curves are labeled with the corresponding value of σ_I^2.

Postfilters

- As we will see in this section, the performance of the MVDR beamformer can be enhanced by applying a frequency dependent weighting to the output of the beamformer.
- This has the effect of introducing a final filtering operation, or a *postfilter*, on the beamformer output.
- Let us again consider the same single plane-wave model as in (2) and (1), and once more assume $F(\omega)$ and $\mathbf{N}(\omega)$ are uncorrelated.
- The spatial spectral matrix of $\mathbf{X}(\omega)$ can be expressed as

$$\boldsymbol{\Sigma}_{\mathbf{X}}(\omega) = \Sigma_F(\omega)\,\mathbf{v}(\mathbf{k}_s)\,\mathbf{v}^H(\mathbf{k}_s) + \boldsymbol{\Sigma}_{\mathbf{N}}(\omega).$$

- Let $D(\omega)$ denote the snapshot of the desired signal, which is equivalent to the source snapshot $F(\omega)$.

Mean-Square Error

- We now define the matrix processor

$$\hat{D}(\omega) = \mathbf{w}^H(\omega)\,\mathbf{X}(\omega).$$

- The *mean-square error* (MSE) is defined as

$$\begin{aligned}\zeta(\mathbf{w}(\omega)) &\triangleq E\left\{\left|D(\omega) - \mathbf{w}^H(\omega)\,\mathbf{X}(\omega)\right|^2\right\} \\ &= E\left\{(D(\omega) - \mathbf{w}^H(\omega)\,\mathbf{X}(\omega))(D^*(\omega) - \mathbf{X}^H(\omega)\,\mathbf{w}(\omega))\right\}.\end{aligned}$$

- In order to minimize the MSE, we take the complex gradient of ζ with respect to $\mathbf{w}(\omega)$ and equate the result to zero, to find

$$\mathcal{E}\left\{D(\omega)\,\mathbf{X}^H(\omega)\right\} - \mathbf{w}^H(\omega)\,E\left\{\mathbf{X}(\omega)\mathbf{X}^H(\omega)\right\} = \mathbf{0},$$

so that

$$\mathbf{\Sigma}_{D\,\mathbf{X}^H}(\omega) = \mathbf{w}^H_{\mathrm{mmse}}(\omega)\,\mathbf{\Sigma}_{\mathbf{X}}(\omega).$$

MMSE Solution

- Hence, the MMSE solution is

$$\mathbf{w}_{\text{mmse}}^{H}(\omega) = \mathbf{\Sigma}_{D\mathbf{X}^H}(\omega)\,\mathbf{\Sigma}_{\mathbf{X}}^{-1}(\omega). \tag{31}$$

- From the signal model, and the assumption that noise and signal are uncorrelated we find

$$\mathbf{\Sigma}_{D\mathbf{X}^H}(\omega) = \mathcal{E}\{D(\omega)D^*(\omega)\mathbf{v}^H(\mathbf{k}_s)+D(\omega)\,\mathbf{N}(\omega)\} = \Sigma_F(\omega)\,\mathbf{v}^H(\mathbf{k}_s).$$

- This implies that (31) can be specialized according to

$$\mathbf{w}_{\text{mmse}}^{H}(\omega) = \Sigma_F(\omega)\,\mathbf{v}^H(\mathbf{k}_s)\,\mathbf{\Sigma}_{\mathbf{X}}^{-1}(\omega).$$

- The spatial spectral matrix of the subband snapshot **X** can be expressed as

$$\mathbf{\Sigma}_{\mathbf{X}}(\omega) = \Sigma_F(\omega)\,\mathbf{v}(\mathbf{k}_s)\,\mathbf{v}^H(\mathbf{k}_s) + \mathbf{\Sigma}_{\mathbf{N}}(\omega).$$

Applying the Matrix Inversion Lemma

- By applying the matrix inversion with

$$\mathbf{A} = \mathbf{\Sigma_N}(\omega), \;\; \mathbf{B} = \mathbf{v}(\mathbf{k}_s), \;\; \mathbf{C} = \Sigma_F(\omega), \;\; \mathbf{D} = \mathbf{v}^H(\mathbf{k}_s),$$

whereupon we find,

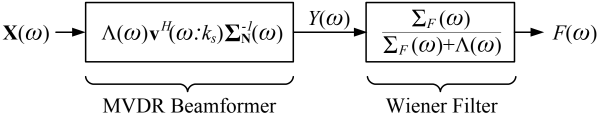
$$\mathbf{\Sigma_X}^{-1} = \mathbf{\Sigma_N}^{-1} - \Sigma_F \, \mathbf{\Sigma_N}^{-1} \, \mathbf{v} \left(1 + \Sigma_F \mathbf{v}^H \, \mathbf{\Sigma_N}^{-1} \, \mathbf{v}\right)^{-1} \mathbf{v}^H \mathbf{\Sigma_N}^{-1}. \quad (32)$$

- Defining $\Lambda(\omega)$ as in (19) and substituting into (32), we learn

$$\mathbf{w}^H_{\text{mmse}}(\omega) = \frac{\Sigma_F(\omega)}{\Sigma_F(\omega) + \Lambda(\omega)} \cdot \Lambda(\omega) \, \mathbf{v}^H(\mathbf{k}_s) \, \mathbf{\Sigma_N}^{-1}(\omega). \qquad (33)$$

Relation between MVDR and MMSE Processors

- Frome (18) and (33), the MMSE beamformer is clearly a MVDR beamformer followed by a frequency-dependent scalar multiplicative factor.
- The multiplicative factor is equivalent to a Wiener postfilter.
- Recall now that $\Sigma_F(\omega)$ is the power spectral density of the signal at the input of the beamformer, which, due to the distortionless constraint (16), is also the power spectral density of the signal at the output of the MVDR beamformer.
- The MMSE beamformer is shown in the figure.



Designing Practical MMSE Processors

- While (33) is optimal in the mean square sense, it is not sufficient to design a MMSE beamformer.
- This follows from the fact that the spectra of both the desired signal $D(\omega)$ and disturbance $\Lambda(\omega)$ at the output of the beamformer must be *known*.
- In practice they can only be estimated, and forming this estimate is the art in Wiener postfilter design.
- One of the earliest and best-known proposals for estimating these quantities was by Zelinski (1988).

Summary

- In this lecture, we considered the conventional adaptive beamforming algorithms.
- These algorithms are based on the notion of minimizing the output power of the array subject to a distortionless constraint.
- We have compared the performance of the conventional adaptive algorithms to the simple delay-and-sum design.
- We also discussed how beamforming performance could be improved through the addition of a postfilter at the output of the beamformer.